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Research Article

How Can We Visualize Mathematical Concepts Outside the Box? Tracing the Journey from Formal Definitions to Inventive "Pictures of Mathematical Concepts"

Matematiksel Kavramları Alışılmışın Dışında Nasıl Görselleştirebiliriz? Formal Tanımlardan Yaratıcı "Matematiksel Kavram Resimlerine" Giden Yolculuğun İzlenmesi

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Abstract

Purpose: *Visualization offers the opportunity to represent the features of mathematical concepts and the relationships between mathematical concepts in various forms. This study aimed to examine the visualization process of mathematical concepts of preservice mathematics teachers (PMTs).* ***Method:*** *The study employed a qualitative method. All participants were given formal definitions of mathematical concepts and asked to draw pictures of them. They attempted to visualize the features and relationships of concepts with the help of drawing and painting. Discussions were carried out in the classroom about whether the pictures were appropriate in terms of representing the concepts visually. Following these discussions, interviews were conducted with the groups whose pictures were deemed suitable by the class to represent the concept visually. The pictures and views of the PMTs regarding the visualization process of the concepts were analyzed through the techniques of the grounded theory method.* ***Findings:*** *It was determined that the PMTs did not randomly draw pictures of the concept; instead, they drew their pictures from the ideas they associated with the concepts. It was also determined that PMTs tend to produce pictures that illustrate more than one concept in one representation, and they think it is more useful this way. It was observed that there were some differences between the pictures of the concepts drawn by the PMTs and their explanations of the concepts, and that PMTs sometimes depicted more or fewer aspects of the concepts than they had thought.* ***Conclusion:*** *Through their pictures of*

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concepts, the PMTs discovered relationships among concepts, or between concepts and pictures, and made some generalizations.

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Matematik öğretmen adayları

Öz

Amaç: *Görselleştirme, matematiksel kavramların özelliklerini ve matematiksel kavramlar arasındaki ilişkileri çeşitli biçimlerde temsil etme fırsatı sunar. Bu çalışma, matematik öğretmen adaylarının (MÖA) matematiksel kavramları görselleştirme süreçlerini incelemeyi amaçlamaktadır.* ***Yöntem:*** *Çalışmada nitel araştırma yöntemi kullanılmıştır. Tüm katılımcılara matematiksel kavramların formal tanımları verilmiş ve bu kavramların resimlerini çizmeleri istenmiştir. Katılımcılar, çizgi ve renklerden yararlanarak kavramların özelliklerini ve kavramlar arasındaki ilişkileri görselleştirmeye çalışmışlardır. Çizilen resimlerin kavramları görsel olarak temsil etme açısından uygun olup olmadığına ilişkin sınıf içinde tartışmalar yapılmıştır. Bu tartışmaların ardından, sınıf tarafından kavramı görsel olarak temsil etmede uygun bulunan resimleri çizen gruplarla görüşmeler gerçekleştirilmiştir. MÖA'ların kavramları görselleştirme sürecine ilişkin resimleri ve açıklamaları, gömülü teorinin teknikleri kullanılarak analiz edilmiştir.* ***Bulgular:*** *Çalışma sonucunda MÖA'ların kavram resimlerini rastgele çizmedikleri, aksine kavramlarla ilişkilendirdikleri düşüncelerden hareketle çizimler yaptıkları belirlenmiştir. Ayrıca MÖA'ların tek bir resimde birden fazla kavramı temsil etme eğiliminde oldukları ve kavramları bu şekilde temsil etmenin daha faydalı olduğunu düşündükleri belirlenmiştir. MÖA'ların kavramlarla ilgili diğer resimleri ya da açıklamaları ile çizdikleri kavram resimleri arasında bazı farklılıkların olduğu; kavram resimlerinde düşündüklerinden daha fazla ya da daha az yönü resmettikleri gözlemlenmiştir.* ***Sonuç:*** *MÖA'lar çizdikleri kavram resimleri aracılığıyla kavramlar arasında ya da kavramlar ile resimler arasında ilişkiler keşfetmiş ve bazı genellemeler yapmışlardır.*

Introduction

Visualization is a powerful tool for making sense of mathematical concepts and their relationships (Rösken & Rolka, 2006) and is generally considered helpful in supporting concept formation (Dreyfus, 1991). Mathematical concepts have a wealth of visual relations that can be intuitively represented in various ways (Guzman, 2002) and declaratively encoded in visual representations without propositions (Liang & Sedig, 2010b). However, mathematical concepts are mainly expressed through formal definitions. Students may not understand the meaning of this formal definition. Visualization also helps grasp formal definitions' hidden meanings (Furinghetti et al., 2011). Thus, visualization can also help clarify unclear points within the definition of the concept. Visualization of mathematical concepts can help reveal potential visual relationships and rediscover concepts with different visual representations. Different visual representations of the same mathematical concept can provide a broad perspective.

Stylianou (2002) stated that mathematicians examine or create visual representations to investigate information in depth, and mathematics teachers should

encourage these behaviors in their students. To encourage their students in these behaviors, mathematics teachers should go through the same experience process. Otherwise, they may not be able to guide their students sufficiently, or they may mislead them. Therefore, preservice mathematics teachers (PMTs) should first be encouraged to create their own mathematical visual representations and make mathematical discoveries.

This study asked PMTs to visualize mathematical concepts by examining their formal definitions. The PMTs have encountered the concepts related to the subdomains of sets and functions in the present study and their visual representations (such as Venn diagrams and function graphs) numerous times throughout their educational lives. Drawing pictures is a natural activity that comes from childhood. Visualizing mathematical concepts by drawing pictures can combine two components familiar to PMTs. This study attempts to reveal how PMTs represent concepts by drawing pictures, how visual representations of concepts they already know affect PMTs' visualization processes, and what criteria PMTs consider when drawing pictures of concepts. This study will contribute to the literature regarding visualization by drawing pictures in mathematics education, drawing the pictures of concepts, and being able to represent concepts from one's own perspective outside of known visual representations.

Visualization

"Visualization generally refers to the ability to represent, transform, generate, communicate, document, and reflect on visual information" (Hershkowitz et al., 1989, p. 75). In visualization, a visual representation of a mathematical problem, concept, principle, or property is created or used (Zimmerman & Cunningham, 1991). Visualization is one type of reasoning (Gutiérrez, 1996). Mathematicians value visualization and try to use it (Borwein & Jorgenson, 2001). It has long been recognized that visualization plays a key role in the work of mathematicians (Noss et al., 1997). Visualization is regarded as an essential means for fostering conceptual understanding in mathematics education. Tappenden (2005) stated that visualization helps eliminate difficulties in learning and teaching mathematics. On the other hand, mathematicians can understand symbolic representations, even if they are very complex. Visualization may be secondary for mathematicians, but it should not be considered this way in mathematics education (Borba & Villarreal, 2005). Visualization is a technique by which abstract concepts of mathematics are made concrete. Through visualization, students can comprehend concrete representations and develop deeper understandings of abstract concepts. Visualization is also a technique in which the concrete is made abstract so that students can apply mathematics and obtain important results for a concrete situation. That is, visualization enables students to make the concrete abstract and the abstract concrete. Thus, visualization allows the creation of both educationally desirable and achievable goals (Macnab et al., 2012). For this to happen, first of all, mathematics teachers need to be able to use visualization effectively. Without the guidance of mathematics teachers, it may not be possible for students to use visualization effectively

in mathematics education. Therefore, PMTs should initially receive training on visualization during their undergraduate studies. Examining the visualization processes of PMTs regarding fundamental mathematical concepts, without any prior training, may provide valuable insights for the design of such instruction. The primary aim of this study is to investigate this process.

Visual Representation

Visual representation is "a term used to describe the representation of a problem situation in such a way that relationships can be visualized, or seen by the eye" (George, 1997, p. 16). Visual representations are frequently used to support mathematical thinking. Visualization of data in statistics, graphs of functions in algebra, and schematic drawings in geometry are examples of visual representations. Examples of visual representations are limitless (Rau & Matthews, 2017). Pictures of mathematical concepts in this study were considered as a new visual representation.

Mathematics educators recommend that visual representations become a natural element of mathematics teaching and be used more in mathematics teaching at different levels of education, especially in middle school, high school, and university (Gutiérrez, 1996). The National Council of Teachers of Mathematics (NCTM, 2000) recommends that teachers encourage students to create their representations, and that these representations be shared and discussed within the classroom. NCTM notes that this allows teachers to initiate class discussions about representing knowledge, assess students' understanding, understand misconceptions about representing knowledge, and create new learning and teaching situations. In this study, the pictures of concepts drawn by the PMTs were shared in the classroom, and other PMTs discussed the pictures of concepts as visual representations. The nature of teachers' understanding of representation affects students' understanding of representations and the development of their skills in using representations (Stylianou, 2010). In this context, having an idea about teachers' understanding of representation can provide a perspective on how students' skills in using representations can be improved. This study can provide insight into teachers' understanding of visual representation.

Pictures of Mathematical Concepts

This section emphasizes the meanings of the mathematical concept and picture terms, and each is explained under separate headings. Considering these explanations, the picture of the mathematical concept is defined.

Mathematical concept

A mathematical concept is "a researcher's articulation of intended or inferred student knowledge of the logical necessity involved in a particular mathematical relationship" (Simon, 2017, p. 123). Mathematical concepts concern mathematical (mental) objects and the relationships between these objects (Simon, 2017). Mathematical objects, on the other hand, consist of logical necessities and the consequences of relationships between

mathematical objects and other mathematical objects. The properties of mathematical objects may be challenging to discover, prove, or disprove (Sierpinska, 1994).

Picture

In the Sanat Kavram ve Terimleri Sözlüğü [Dictionary of Art Concepts and Terms], *picture* is defined as "any two-dimensional figure" and "any two-dimensional composition created on a surface" (Sözen & Tanyeli, 2017, p. 201). *Composition* is defined in the same dictionary as "the bringing together of the elements that make up the work within certain order relations and the work itself that emerges as a result of this work" (Sözen & Tanyeli, 2017, p. 135). Considering the definitions of picture and composition, it can be concluded that the picture should be two-dimensional, that there is no limitation regarding the subject of the picture, and that the elements in the picture should be together in a certain order. In the *Resim Sanatı Sözlüğü*, *a picture* is defined as designing the objects and events of reality with the help of colors on a solid surface (Eroğlu, 2006, p. 298). Based on this definition, it can be said that colors are a factor in the picture and that the picture is the result of design. In the *Sanat Kavram ve Terimleri Sözlüğü*, the English equivalent of picture is "painting, drawing" (Sözen & Tanyeli, 2017, p. 201). In other words, it is mentioned that a picture can be used in both "painting" and "drawing" meanings in English. In English, painting means "a painted picture" (Longman, 2008, p. 721) or "a work of art made with paint" (Webster, 2009, p. 487). Drawing is used in the meaning of "a picture you make with a pen or pencil" (Longman, 2008, p. 306) or "to create a picture of by making lines on a surface" (Webster, 2009, p. 213). As you can see, "painting" and "drawing" are both pictures. However, when it comes to "drawing," the picture is created with lines only, while when it comes to "painting," colors are also included in the picture. In English dictionaries, the word picture has a broader scope and includes the meanings of "painting" and "drawing" (Longman, 2008, p. 753; Webster, 2009, p. 507). In his book Understanding Pictures, Lopes (1996) defines the concept of "picture" as "a representation that embodies information on the basis of which its source can be identified by a suitably equipped perceiver" (p. 151). Therefore, the picture is based on information. In fact, by creating a picture by a suitable perceiver, the information entity turns into a perceptible representation.

Picture of a mathematical concept

Since mathematical concepts are accessed exclusively through representations, they are critical in constructing students' conceptual understanding processes (Duval, 2006; Goldin & Shteingold, 2001). The ability to select an appropriate representation of a mathematical concept and to leverage the strengths of a given representation is a critical element in understanding mathematical ideas (Lesh et al., 1987). Mathematical concepts can be represented in various ways (Duncan, 2010). For example, symbolic expressions, written words, graphical displays, numerals, diagrams, and drawings can be used (Stylianou, 2010). In this study, PMTs attempted to visually represent certain mathematical concepts related to sets and functions. These representations are made on paper; that is, they are two-dimensional. The representation of the concept is made by

considering the definition of the concept. PMTs attempt to bring together the elements in the definition. These elements are arranged in a particular order from the PMTs' perspectives, indicating the presence of composition. Concepts are represented by PMTs using either lines alone or colors together with lines; in other words, concept representations involve "drawing" and "painting". The representation of the concept is derived from the information extracted from its definition. As suitably equipped perceivers, PMTs create representations embodying this information. Therefore, in this study, the representations of the concepts created by PMTs are called the *pictures of mathematical concepts*. "Visual mathematical representations (e.g., geometric structures, graphs, and diagrams) refer to graphical representations that visually encode properties and relationships of mathematical structures and concepts" (Sedig & Sumner, 2006, p.1). *A picture of a mathematical concept is a representation in which the relationships or properties in the concept are shown with drawings or colors.* That is, the picture of the mathematical concept is a new visual mathematical representation.

The Present Study

Visualizations have been used extensively to assist students, with many studies confirming the benefits of visualization for reasoning and learning. Visualizations are intended as concrete tools that allow students to explore more complex mathematical concepts (Liang & Sedig, 2010a). The idea that there is a relationship between mathematical visualization and understanding of concepts is supported by researchers (Trigueros & Martínez-Planell, 2010). However, students fail to see the connections between the formal definition of a mathematical concept and its visual representation in determining whether a representation is an appropriate representation of the mathematical concept (Alcock & Weber, 2008, 2010). While students have difficulty understanding visual representations, mathematicians can discover new information through visualization (Arcavi, 2003; Bråting & Pejlare, 2008). While students have difficulty understanding visual representations, mathematicians can discover new information through visualization (Arcavi, 2003; Bråting & Pejlare, 2008). Such difficulty arises because visualization does not directly provide ready-made information; individuals require experience and perspective to identify and uncover mathematical connections. Therefore, students need to be trained in mathematical visualization to develop a perspective similar to that of mathematicians. Students need support from their teachers in understanding mathematical concepts. In this context, teacher training comes to the fore, and PMTs should be provided with sufficient experience in visualization. There is a need for research to be conducted with PMTs to guide the development of training programs on visualization.

Individuals who engage in mathematics and those who learn mathematics encounter significantly different experiences when utilizing visualization in mathematics. In this context, how does the same situation apply to mathematics teachers? Investigating how PMTs express mathematical concepts through the visual representations they create, similar to mathematicians, can provide valuable insights into this question. Answers to

questions such as how a connection can be established between the formal definition of the concept and a visual representation, the difficulties encountered while trying to establish these connections, and how a solution can be produced against these difficulties can help overcome the problems encountered in visualization. Investigating how PMTs create their own visual representations by considering the formal definition of concepts may provide clues about the answers to these questions. However, to date, no studies in the existing literature examine the visualization processes of PMTs' concepts of sets and functions. Therefore, this study aims to examine the visualization processes of mathematical concepts related to the sets and functions learning sub-domains of PMTs. In line with this general aim, the study's research questions are as follows:

1) How does the process of creating pictures of mathematical concepts develop in PMTs?
2) How do PMTs analyze the pictures they draw of mathematical concepts?

Methodology

This study employed a qualitative methodology to examine the visualization process of mathematical concepts and the meanings of the pictures that emerged from this process. One reason for choosing a qualitative approach is its ability to better understand the visualization processes of mathematical concepts from the participants' perspectives within their natural environment. Furthermore, this methodology was selected to interpret the challenges and factors that participants encounter while visualizing concepts, as these can be best understood through their perspectives (Denzin & Lincoln, 2005; Merriam & Tisdell, 2015). Another reason for using a qualitative approach is the lack of substantial information about the research phenomenon in the literature (Creswell, 2012).

Participants

This study was conducted with thirty-four PMTs enrolled in an elective course offered by the Department of Mathematics Education at a state university in Türkiye. Before the study, the researcher provided the PMTs with detailed information about the process. PMTs who were interested and agreed to participate voluntarily were selected based on their enrollment in the elective course conducted by the researcher. The participants had previously taken the following undergraduate courses: Analysis I, II, and III; Fundamentals of Mathematics I and II; History of Mathematics; Abstract Mathematics; Analytical Geometry; and Linear Algebra I.

Tasks

The PMTs draw pictures of concepts by examining the formal definitions of the concepts (see Supplementary Information (SI) 1) related to the sub-learning areas of sets and functions.

Data Collection

In the study, the researcher continuously observed the PMTs' visualization of concepts. In addition, interviews were conducted with the PMTs regarding their concept visualization processes and their pictures of concepts. The interviews were video recorded with the consent of the PMTs. A document analysis was carried out on the pictures of concepts the PMTs drew. The study documents consist of all the drawings made by the PMTs related to mathematical concepts. In order to reflect the PMTs' processes of visualizing concepts, they were asked not to use erasers while drawing and to submit all of their drawings. During this process, the researcher did not intervene. The drawings produced by the PMTs were analyzed in the order in which they were created. In this way, the aim was to understand the PMTs' processes of visualizing concepts and to reveal, step by step, how they drew pictures related to the concepts.

Procedure

The study covers the work of the PMTs on the sub-learning areas of sets and functions during weeks 10, 11, and 12 of a 12-week elective course offered by the Department of Mathematics Education at a public university. Since the number of concepts examined in the elective course was relatively high, only the concepts visualized during the last three weeks were included in the study. The flow of the lesson was negatively affected during the first week due to the lengthy process of creating the pictures of concepts and the overcrowded class. In response to the requests from the PMTs, the course was divided into two sections, one with 20 PMTs and the other with 14. The course was held in two sections and on separate days in the remaining weeks. In the lessons, the researcher first wrote the formal definitions of mathematical concepts on the board and asked the PMTs to visualize these concepts by drawing pictures. Based on the formal definitions provided, these visualizations were to differ from previously encountered representations.

The PMTs were not trained to create pictures of the concepts because the research focused on how PMTs visualized the concepts according to their own perspectives. During the course, the PMTs worked in groups of two or three or individually, depending on their preferences, while the researcher observed their work. In the later weeks of the course, it was observed that some PMTs began to prefer working individually rather than in groups. The PMTs were asked the reason for this situation. They stated that during the initial weeks of the course, they preferred group work because they did not know how to draw pictures. In the following weeks, however, they noted that a wide range of ideas emerged within the group regarding how to visualize the concept. For this reason, the group members preferred to work individually in order to reflect on their own ideas. The presentations began once all the PMTs had completed their pictures of concepts in the lessons. The PMTs drew their pictures of concepts on the board, explained them to the class, and the class discussed these representations in terms of their appropriateness for representing the concepts. The class evaluated all pictures of the concepts; within this process, consensus was defined as the absence of any substantial opposing view following classroom discussions and the emergence of a class-wide shared agreement that the representation appropriately reflected the concept. The researcher identified the pictures

for which a clear consensus had emerged by observing these discussions. As the process progressed, it was observed that the class reached a shared judgment more quickly and consistently when evaluating the pictures of concepts. This evaluative process became more apparent during the final three weeks of the course**.** The duration of the lessons was flexible, averaging 6 hours per lesson in both sections. After the lessons, the interview participants were selected using criterion-based purposive sampling, and only those PMTs who had reached a classroom consensus on the appropriateness of their visual representations of the concept, whether working in groups or individually, were included in the interviews. The interviews were conducted at the end of the lessons, and the composition of interview participants (whether working in groups or individually) varied each week. In this way, the process by which PMTs created pictures of concepts was examined through representations that had been discussed and accepted within the classroom context. The researcher conducted interviews with participants to explore the factors they considered when visualizing the concept, how they approached the task, how they decided on the picture, the challenges they encountered, and the significance of the lines and colors in the pictures of concepts they created.

Data Analysis

Since the study aimed to examine the processes of visualizing mathematical concepts of PMTs and the connections and properties within the pictures of concepts, the data were analyzed using open, axial, and selective coding techniques derived from grounded theory (Cohen et al., 2011). The names of the PMTs were not used in the analyses, and coding was done for each group and PMT. For example, in the codes $G_5M_1M_2$ and G_6M, G_5 and G_6 represent the group and its number in the study (with G_5 indicating the 5th group and G_6 indicating the 6th group). In contrast, M_1M_2 and M indicate the number of PMTs in each group (M_1M_2 denotes two PMTs, and M denotes one PMT). The researcher reviewed the transcripts of the video recordings and concept-related pictures of each interviewed group multiple times. In this context, the data were coded, notes were created regarding the data, and diagrams were drawn. First, open coding was conducted to create categories. During this process, the data were disassembled, examined, compared, conceptualized, and categorized. Then, efforts were made to establish relationships between categories and subcategories and reorganize the data fragmented by open coding, known as axial coding. Afterward, selective coding was performed to select the categories. In the selective coding phase, main categories were selected by systematically developing categories associated with other categories, confirming these relationships, and requiring further development or improvement (Punch, 2000). During the analysis, efforts were made to follow the sequence in which the categories emerged, and notes were taken throughout this process. Relationships between categories, subcategories, and dimensions were created by examining these notes. Each time the codes were reviewed, the researcher first read the sentence, tried to predict which codes it might belong to, and then examined the assigned category names and their descriptions. Thus, consistency with the previous coding was checked. Each theoretical code that emerged from the analysis consists of representative drawings and explanations of PMTs.

Results

This section describes the processes by which participants, who attempted to visualize different concepts over 12 weeks, visualized several mathematical concepts related to the learning domains of sets and functions in Weeks 10, 11, and 12. The descriptions summarize the core content of the emergent categories, whereas the PMTs' drawings and their statements about how they thought during the drawing processes are provided in the SI due to the length of the manuscript.

The Category of Thinking About the Relationship Between Concept and Picture

This category includes the thinking process that the PMTs went through while visualizing mathematical concepts, which focused on the concept itself, the construction of the picture, and the relationships between the two. Table 1 summarizes the subcategories, dimensions, and subdimensions within this category (see SI 2 for the supporting evidence of the table).

Table 1. *Structure of the Category of Thinking About the Relationship Between Concept and Picture*

Subcategory	Dimension	Sub dimension
Concept: Contains the thoughts of the PMTs about the concept.	Reviewing the definition: Contains the PMTs' own explanations regarding the definitions of the concepts.	
	Sampling: Contains examples determined by the PMTs regarding the concepts.	
	Generating ideas: Includes explanations in which they state that they drew the pictures to get ideas about the concept.	
Picture: Indicates the ideas at the starting point of the created picture of concept and the purpose for which the lines and colors in the picture of concept are used.	Source of inspiration: Involves how PMTs identify a source of inspiration by establishing similarities between their prior knowledge and the pictures they draw when visualizing concepts.	Discovering a source of inspiration: Includes the prior knowledge of PMTs that underlies the background of their pictures.
		Developing the source of inspiration: Refers to the prior knowledge of PMTs that becomes a factor in creating the picture of the concept.
	Determining representation: Refers to PMTs' attributing meanings to some colors and lines to represent the concept while creating their pictures.	Color: PMTs attributed meanings to some colors while creating their pictures.
		Line: PMTs attributed meaning to some lines while creating their pictures.
Exploring a new situation: Refers to the situations in which the PMTs reach generalizations between concepts or between the concept and the picture while drawing their pictures.	Concept-picture relationship: Contains generalizations about the concept and picture relationship.	
	Concept-concept relationship: Includes generalizations between concepts.	

PMTs drew their pictures by examining the formal definitions of the concepts. In describing these pictures, they talked about their process of examining the definitions and

made explanations about them. They determined examples related to the concepts and constructed their pictures based on these examples. It was observed that examples appeared in all the pictures of concepts. They attempted to visualize the concepts, and some pictures were intended to support their understanding of the concept through drawing. Their prior knowledge provided them with various perspectives when visualizing the concepts, so they tried to draw their pictures by establishing connections between what they already knew and the concepts. In certain instances, this information remained just a sketched idea, and sometimes, it helped them draw a picture of the concept. They determined representations through colors and through lines during the drawing process. They drew various conclusions about the relationship between the pictures and the concepts from these drawings. They discovered the properties that concepts should have through the pictures they drew.

The Category of Drawing a Picture

This category includes all the drawings and colorings created by the PMTs to visualize the concepts, that is, all the pictures they produced. Table 2 summarizes the subcategories and dimensions within this category (see SI 3 for the supporting evidence of the table).

Table 2. *Structure of the Category of Drawing a Picture*

Subcategory	Dimension
Familiar picture: Represents the drawings that PMTs already know about the visual representation of concepts.	Venn diagram
	Function graph
	Coordinate system
Concrete picture: Contains pictures of real beings found in nature to represent the concept of PMTs.	
Draft picture: These are the pictures that PMTs stated that they drew to experiment while trying to draw a picture of the concepts.	Purposeful draft picture: These are the pictures that PMTs say they drew to try for a particular purpose.
	Purposeless draft picture: These are the pictures that PMTs stated they did not have any thoughts on while drawing.
Mismatched picture: Represents the pictures that differ from PMTs' explanations and other similar pictures. Such mismatched pictures emerge as a result of differences in the act of drawing.	Incomplete drawing: Refers to cases in which PMTs draw their pictures incompletely compared to their explanations and other drawings.
	Excessive drawing: Refers to cases in which PMTs draw their pictures with additional or redundant elements compared to their explanations and other pictures.
Coloring: Contains all colorings of PMTs pictures of concept.	
Abstract picture: Contains pictures of PMTs that do not depict real beings found in nature.	Representing the concept: Some of the abstract pictures of PMTs represent only one concept.
	Representing concepts: Some of the abstract pictures of PMTs represent more than one concept.

While trying to visualize the concepts, the PMTs tried to develop a perspective to draw a picture of the concept by making drawings they had previously learned. They established similarities between concepts and concrete elements and drew pictures representing the concepts based on these similarities. While they tried to draw pictures of the concepts, some were drawn for trial purposes, and some were drawn

unconsciously, in a way different from what they thought. When drawing the pictures of the concepts, they did not rely solely on lines but also drew on the visual possibilities offered by colors. In some of the abstract pictures they drew, they drew a picture of a single concept, whereas in others, they drew pictures of multiple concepts.

The Category of Evaluation of the Picture

This category includes the situation assessments made by the PMTs regarding their pictures. Table 3 summarizes the subcategories, dimensions, and subdimensions within this category (see SI 4 for the supporting evidence of the table).

Table 3. *Structure of the Category of Evaluation of the Picture*

Subcategory	Dimension	Subdimension
Evaluation of the picture itself: Contains the situational assessments made regarding the picture itself.	Representation: Contains statements that PMTs consider the picture sufficient or insufficient regarding representation.	Insufficient
		Sufficient
	Design: Contains statements that PMTs consider the picture sufficient or insufficient regarding design.	Insufficient
		Sufficient
Evaluation by comparing with another picture: Refers to comparing the pictures drawn by the PMTs with each other.	Design: Includes statements in which PMTs evaluate a picture as successful or unsuccessful regarding design compared to another picture they drew.	Unsuccessful
		Successful

The PMTs evaluated the pictures they drew in terms of their representation of the concept. While they considered some of the pictures insufficient to represent the concept, they considered others sufficient. Although they found certain pictures sufficient in terms of representation, they also evaluated the same pictures in terms of their design. The PMTs found that some of the pictures they considered sufficient in terms of representation were insufficient in terms of design, whereas others were deemed sufficient in both respects. They also compared the pictures of the concepts with each other in terms of design and tried to decide which design was more successful.

The Category of Altering the Picture

This category includes the alterations made by the PMTs to their pictures, which emerged in terms of representation, design, and size. Table 4 summarizes the subcategories and dimensions within this category (see SI 5 for the supporting evidence of the table).

Table 4. *Structure of the Category of Altering the Picture*

Subcategory	Dimension
Representation: Includes the alterations PMTs made regarding the representation of concepts in their pictures.	Partially: Includes the partial alterations PMTs made regarding the representation of concepts in their pictures.
	Completely: Includes the complete alterations PMTs made regarding the representation of concepts in their pictures.
Design: Includes the alterations PMTs made regarding design in their pictures.	Partially: Includes the partial alterations PMTs made regarding the design of concepts in their pictures.
	Completely: Includes the complete alterations PMTs made regarding the design of concepts in their pictures.
Size: Includes the alterations PMTs made regarding size in their pictures.	Shrinking: Includes PMTs shrinking the size of their pictures.
	Enlargement: Includes PMTs enlarging the size of their pictures.

After the PMTs evaluated the pictures of concepts in terms of representation and design, they attempted to alter the parts they considered insufficient in their pictures. They partially altered the pictures they considered partially insufficient in terms of representation or design, whereas they completely altered the pictures they considered entirely insufficient. Additionally, they evaluated the pictures of the concepts in terms of size and altered the size of the pictures when necessary.

The Category of Analyzing the Picture

This category includes the analyses made by the PMTs of the meanings expressed by the pictures they drew about the concepts. Table 5 summarizes the subcategories and dimensions within this category.

Table 5. *Structure of the Category of Analyzing of the Picture*

Subcategory	Dimension
Directly: Contains the comments of PMTs without pointing out any concept related to the picture (e.g., Fig. 1).	Examining the picture: Contains explanations for how PMTs express specific points in the picture (e.g., Fig. 1).
	Interpreting the picture: Specifies which concept is represented in terms of meaning in the PMTs' explanations about the picture within the *examining the picture dimension* (e.g., Fig. 1).
According to the concept: Contains the comments of PMTs about their pictures according to a concept they determined (e.g., Fig. 2).	Determining the concept: Specifies which concept the PMTs will use to interpret the picture (e.g., Fig. 2).
	Examining the picture according to the concept: Contains explanations in which the PMTs express specific points in the picture according to the concept they determined (e.g., Fig. 2).
	Interpreting the picture according to the concept: Contains the explanations in which the PMTs, based on their analyses within the *examining the picture according to the concept dimension,* indicate which aspects of the concept are represented in their picture in terms of meaning and which are not (e.g., Fig. 2).

After completing their pictures of the concepts, the PMTs explained what each picture represented conceptually and what kind of meaning it conveyed.

Figure 1. *Analyzing the Picture Category, Directly Subcategory, (a) Examining the Picture Dimension, and (b) Interpreting the Picture Dimension*

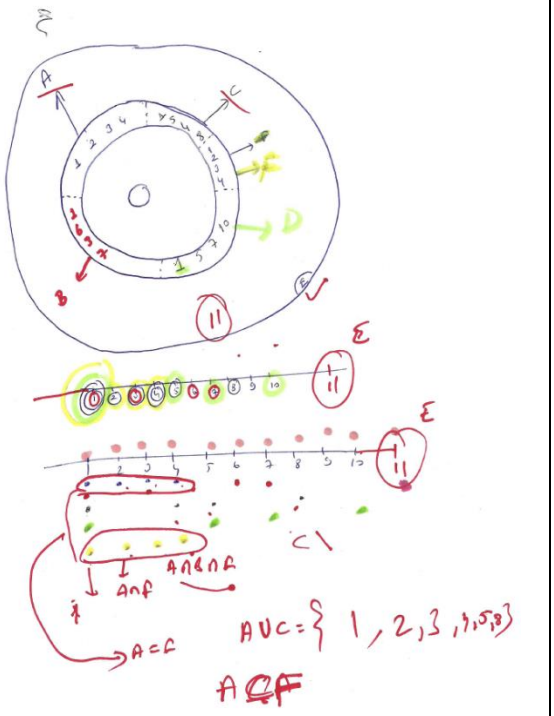
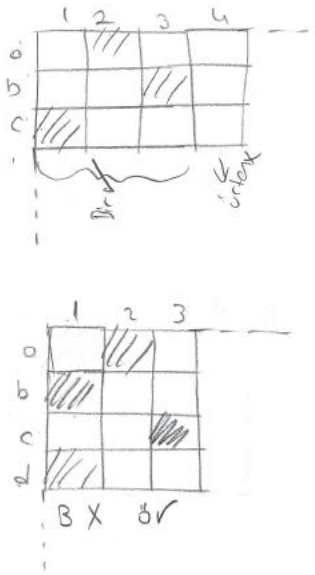
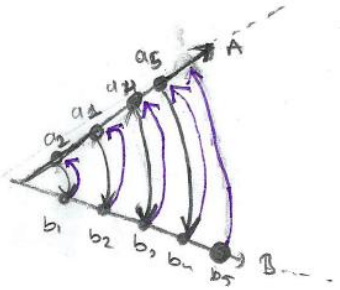
Dialogue between PMTs and R	**Sample documents**
G_2M_1: (**a**) The green set is D, the blue set is A, the black set is C, the red set is B, and the yellow set is F. We see that they all intersect at 1 because each of them has a circle or a dot here. (**b**) Then, we say that 1 is the common element of all the sets here. G_2M_1: (**a**) When we look at 2, there are only blue and yellow dots in 2. When we look here, there are blue and yellow circles. Yellow is set F, blue is set A. (**b**) So, this element here (referring to 2) is located at the intersection of A and F.	

Figure 2. *Analyzing the Picture Category, According to the Concept Subcategory, (a) Determining the Concept Dimension, (b) Examining the Picture According to the Concept Dimension, and (c) Interpreting the Picture According to the Concept Dimension*

Dialogue between PMTs and R	**Sample documents**
G_5M_2: In these pictures, we showed (**a**) an injective function but not a surjective (picture at the top) and (**a**) a surjective function but not an injective (picture at the bottom). R: Ok, how did you draw them? G_5M_1: (**b**) Here, element a matches with 2; b matches with 3; c matches with 1. 4 in the codomain remains empty. (**c**) Therefore, it is not a surjective function. However, it is a function, injective, too (picture at the top). (**b**) In the other case, b matches with 1; a matches with 2; c matches with 3; d also matches with 1.	

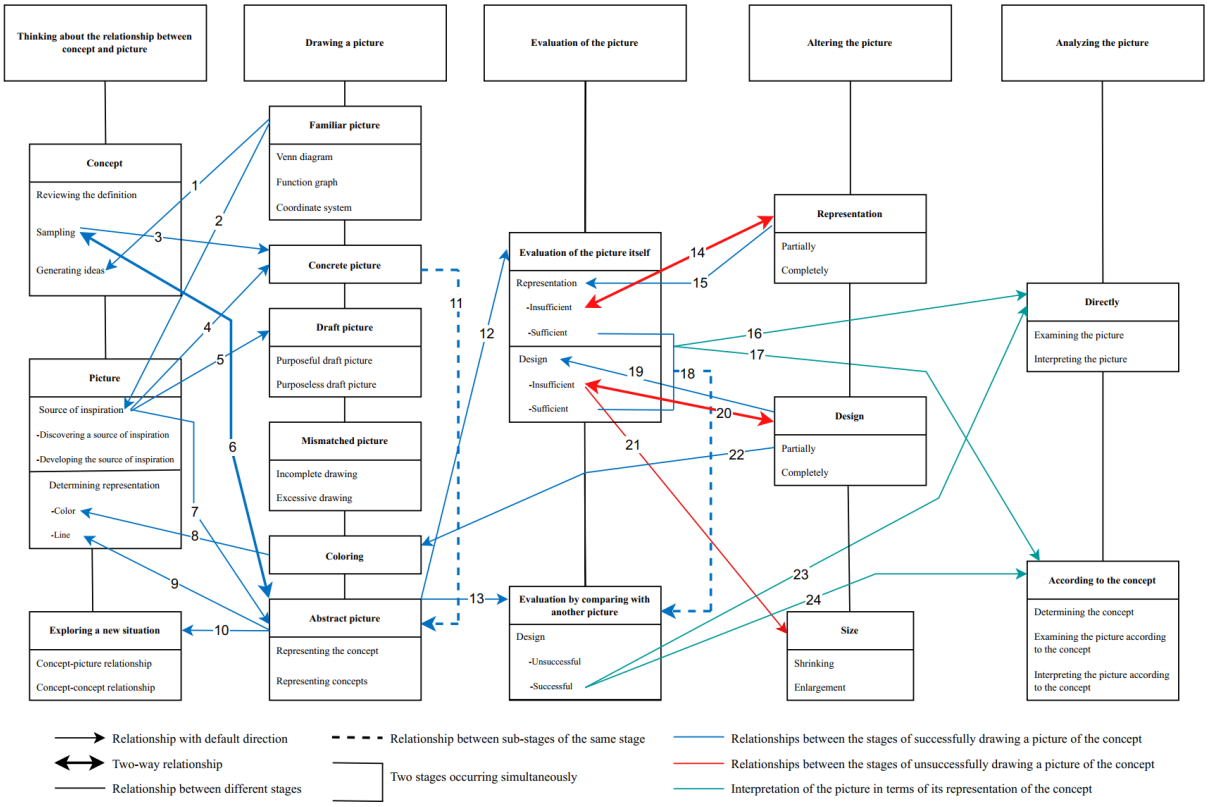
(**c**) So, it is not an injective but a surjective function. Since no empty element exists in the codomain, it is a surjective function (picture at the bottom).	
G_3M: (**a**) Here, I showed the function that does not have an inverse. (**b**) There is no bijective here because we are going from A to B. Since an empty element exists in B, there is no surjective. (**c**) In that case, the inverse of the function is not a function. (**b**) If I were to take the inverse, there is no element that matches with b_5.	

These explanations were sometimes made by directly examining the picture to describe what it meant in terms of the concept (e.g., Fig. 1), and sometimes by first determining the concept and then explaining how it appeared in the picture (e.g., Fig. 2).

Discussion

The visualization processes of certain mathematical concepts in the learning areas of sets and functions of the PMTs obtained in this study are presented in Figure 3, and arrows indicate the relationships between the stages.

Figure 3. *The Process of PMTs Visualizing Certain Mathematical Concepts*



Note. The arrows on the figure numbered to help the reader better follow the explanations related to the process; the numbers on the arrows do not serve any other function.

The category of *thinking about the relationship between concept and picture* progresses alongside the category of *drawing a picture*, as it reflects the underlying thoughts behind the pictures drawn by the PMTs. Therefore, more relationships emerged between these two stages compared to the other stages. One-way arrows indicate that the relationship is one-way and point to the stages where the relationship begins and ends. For example, a *familiar picture* (such as a *Venn diagram*, *function graph*, or *coordinate system*) that is known by the PMTs drawn to represent mathematical concepts may have helped them *generate* an *idea* about how to visualize the concept (arrow 1) or *discover a source of inspiration* for visualizing the concept (arrow 2). The PMTs then further *developed* this *source of inspiration* by drawing other pictures (*concrete picture, draft picture, abstract picture*) (arrows 4, 5, and 7).

It was observed that the order of appearance of the examples determined by the PMTs for the concepts to be used in their pictures of concepts was different. Some PMTs first exemplified the concept and then tried to draw pictures related to the concept based on these examples (arrows 3 and 6). Other PMTs first tried to draw a picture representing the concept and then placed examples related to it in the picture they thought represented it. Therefore, a bidirectional relationship emerged, which was represented by a two-way arrow and a thicker line (arrow 6).

PMTs generally began drawing *familiar picture*s (*Venn diagram, function graph, coordinate system*) about the representation of concepts. They looked for ways to visualize the concepts differently by thinking about these pictures. It was observed that the pictures drawn by PMTs also included *concrete picture*s. In other words, the PMTs established connections between concepts and real-world objects that existed in nature according to their own perspectives and attempted to represent concepts using these connections. By relating concepts to examples from mathematics or real-world objects existing in nature, their efforts to draw pictures might have encouraged them to question their prior knowledge and form connections between their previous knowledge and conceptual knowledge. Thus, while trying to draw pictures of concepts, the PMTs may have also gained different perspectives.

Not all of the pictures drawn by PMTs were intended to represent concepts. It was observed that PMTs sometimes drew *draft picture*s to form ideas about the concept, test their thoughts (*purposeful draft picture*), or did not have any specific purpose (*purposeless draft picture*). This is because visualizing a concept requires having knowledge about the concept, forming ideas based on this knowledge, and designing how to reflect these ideas in a picture. Therefore, it was observed that PMTs primarily drew pictures to understand concepts (*draft picture*). One reason for drawing these pictures could be their unfamiliarity with thinking this way about the concept beforehand. The process of drawing pictures of concepts provided PMTs with the opportunity to examine the concepts from a different perspective during this time.

It was observed that the pictures of the PMTs did not always match each other. PMTs sometimes drew pictures that differed from similar pictures or their explanations about the pictures (*mismatched picture*). Their pictures may have deficiencies (*incomplete drawing*) or redundancies (*excessive drawing*). It was observed that some PMTs were not aware of this situation or did not care about it. PMTs also used colors (*coloring*) to represent the concepts. While coloring, they considered the relationship between the concept and the picture. The colors in the picture represented the concept's elements or properties according to the PMTs' perspectives (arrow 8).

In their abstract pictures, PMTs sometimes represented relationships or properties within concepts using lines according to their own perspectives (arrow 9). Some PMTs visualized the same concepts first through concrete pictures and then through abstract ones. In this case, since a relationship emerged between different subcategories (*concrete picture* and *abstract picture*) within the same category (*drawing a picture*), their relationship is represented with dashed lines rather than a solid line (arrow 11).

The PMTs drew pictures of concepts representing one or more concepts. In the pictures of concepts representing multiple concepts, they could see various properties and relationships related to the concepts together. Indeed, Furinghetti et al. (2011) stated that while a single condition can be checked at once in a symbolic approach, visualization allows for simultaneously testing multiple hypotheses and provides a general idea about

the conditions. The PMTs also indicated that they consciously drew pictures of concepts representing multiple concepts, believing that such pictures would be more helpful.

When PMTs drew the pictures of concepts (*abstract pictures*), they made generalizations about *concept–picture relationship* and *concept–concept relationship*, and as a result, they explored *a new situation* (arrow 10). They discovered the relationships and properties of the concepts in the pictures they drew and expressed that they could easily see the reflection of certain properties of the concept in these pictures. When drawing the pictures of concepts, the PMTs also had the opportunity to question the properties of the concepts and to observe the relationships between them. Researchers have noted that students who make their own drawings understand mathematical operations better than those who use only symbols or observe drawings made by others (Crespo & Kyriakides, 2007). For example, through their pictures of concepts, the PMTs questioned the necessary properties for a function to have an inverse (i.e., the function must be injective and surjective) and stated that they had better understood the concept of the inverse function. In this context, drawing pictures of concepts can help PMTs learn and teach mathematical concepts more effectively.

The visualization processes of the PMTs revealed that they tried to draw pictures of the concepts by starting from a known visual representation of the concept, real objects existing in nature, or their knowledge in different fields. These ideas constituted the *source of inspiration*. Some of these ideas remained only at the stage of *discovering a source of inspiration* and could not be developed further. In contrast, others guided the PMTs and played an influential role in constructing the picture of the concept, thereby *developing the source of inspiration*. In addition, it was observed that the PMTs did not draw the representations in their pictures randomly. Just as each line and color has a meaning in a picture, the PMTs attributed meanings to the lines and colors in their pictures of concepts, to represent the elements or properties of the concepts.

PMTs made selections from the many pictures they drew to decide whether the picture accurately represented the concept. During the selection process, they evaluated whether the picture reflected the concept through *evaluation of the picture itself* (arrow 12) and *evaluation by comparing with another picture* (arrow 13). PMTs attempted to alter pictures they considered *insufficient* regarding *representation* or *design*, either *partially* or *completely* (arrows 14, 20, and 21). In cases where they considered their pictures *partially* insufficient in *design*, they added *coloring* to them (arrow 22). PMTs considered every detail of their pictures and also evaluated their *size*. If they considered the *size insufficient* from a *design* perspective, they *altered* the picture's dimensions (arrow 21). After altering their pictures, they *evaluated* them regarding *representation* and *design* (arrows 14, 15, 19, and 20). If the picture was still *insufficient* regarding *design* or *representation*, they again *altered* it *partially* or *completely* (arrows 14, 20, and 21). PMTs worked towards drawing the picture that best represented the mathematical concept. To achieve this, they avoided unnecessary drawings that they felt did not reflect a property of the concept or a relationship within it, and they *altered* their pictures accordingly (arrows 14, 15, 19, and 20). Macnab et al. (2012) stated that educators should be aware

of the importance of simplicity, as many students seem to struggle with interpreting visual representations. Students may struggle to distinguish relevant information from irrelevant content when visual representations have excessive detail. In this regard, the PMTs' efforts to avoid unnecessary drawings may help make the pictures of concepts more easily understandable for learners. If they considered the pictures they drew *sufficient* in *representation* and *design* (*evaluation of the picture itself*), they then compared the pictures from a design perspective (*evaluation by comparing with another picture*). They judged which picture was more *successful* in representing the concept (arrow 18).

After drawing their pictures of concepts, PMTs analyzed them based on the concepts. Similar to how an artist interprets and explains their completed pictures, PMTs also interpreted their pictures and articulated the meanings conveyed by the pictures about the concepts. PMTs carried out *analyzing the picture* after *evaluating the picture itself* and considering that it was *sufficient* regarding *representation* and *design* (arrows 16 and 17), or after *evaluation by comparing with another picture* they had drawn and considering it was *successful* regarding *design* (arrows 23 and 24). The PMTs made *direct* explanations that drew attention to certain parts of the picture; in other words, they *examined the picture* and expressed the meanings of these explanations regarding the concept (*interpreting the picture*) (arrows 16 and 23). Alternatively, they first *determined the concept* and then *examined the picture according to the concept*. As a result of this examination, they *interpreted the picture according to the concept* (arrows 17 and 24).

Conclusion and Suggestions for Future Research

This study examined how PMTs visualized certain concepts related to sets and functions, that is, how they drew pictures of these concepts. PMTs developed new visual representations of the concepts through the pictures of concepts they drew. The study included interviews with PMTs whose pictures of concepts were considered sufficient visual representations. In contrast, the pictures and thought processes of other PMTs were beyond the scope of the study. Therefore, conducting studies that examine the processes of visualizing concepts, different types of concepts, or the challenges encountered during the visualization process could provide a more detailed perspective.

PMTs stated that although others may not understand the concepts from the pictures of concepts they drew, they themselves could comprehend the concepts directly from their own drawings. Future studies could investigate the effect of participants drawing their own pictures of concepts on their understanding of the concepts, as well as the effect of pictures of concepts drawn by others on students' understanding.

Attempting to represent concepts through drawing encouraged PMTs to think more carefully about the meanings of concepts, to question the relationships and properties of concepts, to recognize their distinguishing features, and to interpret concepts according to their own perspectives. Engaging students across various educational levels in activities that involve drawing pictures of concepts may help them experience similar processes, leading to more profound reflection on concepts, which are the fundamental building blocks of mathematics.

Limitations

In the study, the pictures drawn by the PMTs, either individually or in groups, were discussed by the whole class regarding how well they represented the concept. Therefore, the PMTs evaluated both their own pictures and those drawn by their peers. This practice may have enhanced the PMTs' perspectives on constructing and evaluating the visual representation of concepts. Indeed, toward the final weeks of the elective course, the PMTs were observed to generate more ideas related to the pictures of concepts; moreover, they stated that they preferred working individually rather than in groups to reflect more deeply on their own ideas. The PMTs had not received any formal instruction in visualizing concepts; however, this process itself may have functioned as a form of self-education. In addition, interviews were conducted after class with the PMTs whose pictures were agreed upon in class as visually representing the concept. However, given that the interviews took place after the lessons rather than during the visualization process itself, the nuanced details of how the PMTs visualized the concepts may not have been fully represented in the study. Although all of the formal definitions presented to the PMTs in the study are included in SI 1 (Tables S1 and S2), not all of these definitions were visualized by the PMTs. The corresponding pictures of concepts can be found in Appendices 1 and 2 (Figs. 4 and 5).

Ethical Approval

As the authors of the article, we declare in this form that scientific and ethical rules are followed in this article and that the article does not require the permission of ethical committee for the reason because, at the time the data were collected, formal ethical committee approval was not required according to the institutional regulations in force. Additionally, the first author signed an ethics compliance declaration required by the graduate institute to which the thesis was submitted, affirming adherence to institutional and national ethical standards. All research procedures were designed and implemented in full accordance with the ethical principles of the Declaration of Helsinki (WMA, 2013), as well as with the relevant national legal and institutional guidelines governing research with human participants.

Author Contributions

İlknur Gülşen Turgut: Conceptualization, Methodology, Data Curation, Formal Analysis, Investigation, Visualization, Writing, Review and Editing.

Ahmet Arıkan: Conceptualization, Methodology, Supervision, Review and Editing.

Conflict of Interest

The authors declare no conflict of interest.

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Artificial Intelligence Usage Statement

The authors used GenAI tools to assist with language clarity during the preparation of this study. All content was carefully reviewed and revised, and the authors remain fully responsible for its accuracy and integrity.

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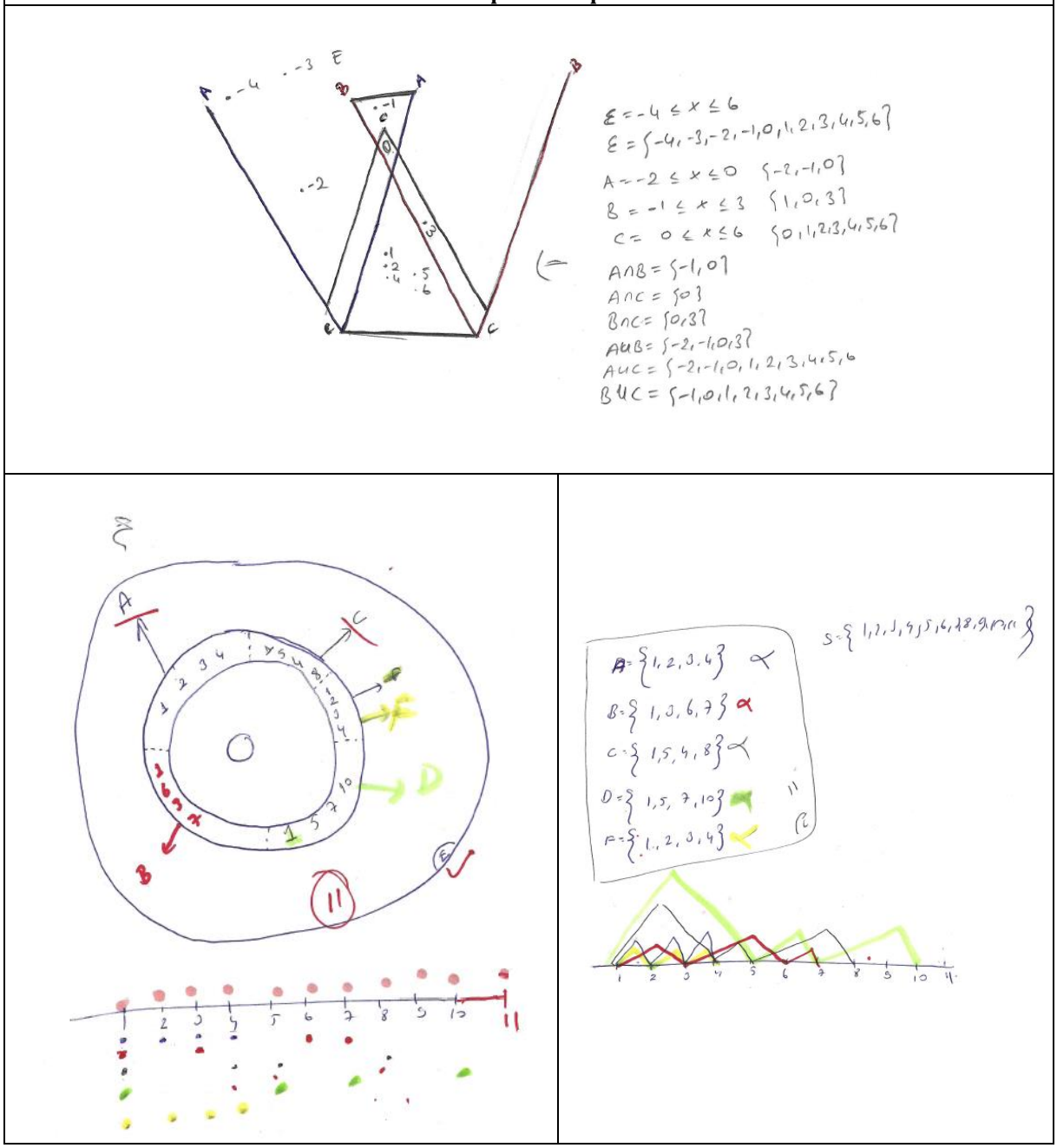
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Appendix 1

Figure 4: *Examples of Pictures of Concepts in the Set Sub-learning Area*

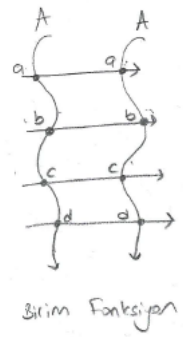
Multiple concepts

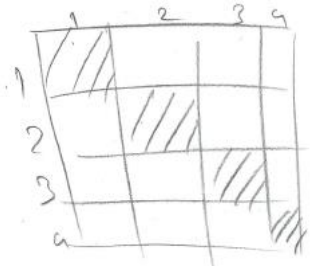


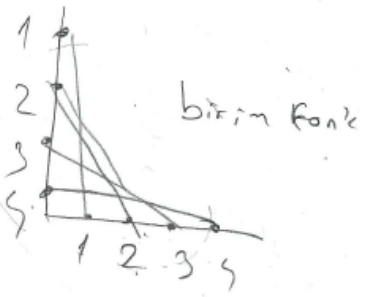
Appendix 2

Figure 5: *Examples of Pictures of Concepts in the Function Sub-learning Area*

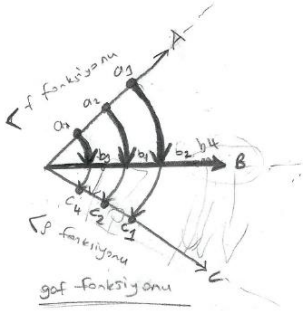
Identity function

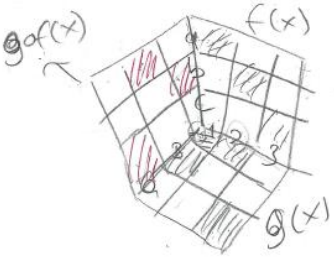


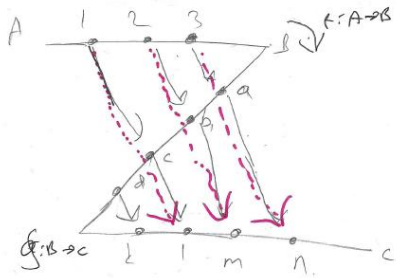




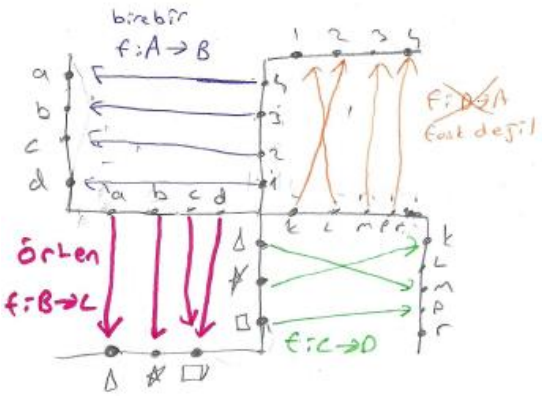
Composition of functions

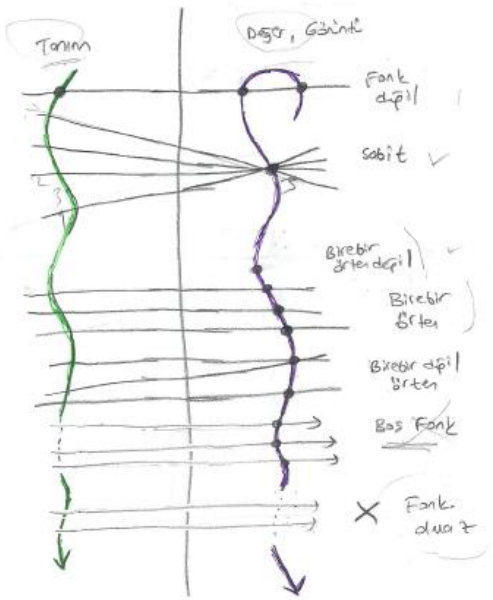






Multiple concepts





Supplementary Information 1:

Formal Definitions Related to Sets and Functions

Table S1. *Formal Definitions of Concepts Related to the Sets Sub-learning Area*

Name of the concept	Definition
Subset	If $\mathcal{E}$ is a universal set and $A, B \subseteq \mathcal{E}$, then $\forall x(x \in A \Rightarrow x \in B)$ A is called a subset of B and is denoted as $A \subseteq B$ or $B \subseteq A$ (Arıkan & Halıcıoğlu, 2018, p. 82).
Equal sets	If A and B are two sets, if $A \subseteq B$ and $B \subseteq A$, then sets A and B are said to be equal and are shown as $A = B$ (Arıkan & Halıcıoğlu, 2018, p. 83).
Empty set	The set that has no elements is called the empty set. The empty set is denoted as $\{\ \}$ or $\emptyset$ (Arıkan & Halıcıoğlu, 2018, p. 83).
Union of sets	The union of sets A and B, where $\mathcal{E}$ is a universal set and $A, B \subseteq \mathcal{E}$, is defined as $A \cup B = \{x \mid x \in A \vee x \in B\}$ (Arıkan & Halıcıoğlu, 2018, p. 91).
Intersection of sets	The intersection of sets A and B, where $\mathcal{E}$ is a universal set and $A, B \subseteq \mathcal{E}$, is defined as $A \cap B = \{x \mid x \in A \wedge x \in B\}$ (Arıkan and Halıcıoğlu, 2018, p. 91).
Absolute complement of a set	Let $\mathcal{E}$ be a universal set and $A \subseteq \mathcal{E}$. Then the set $\{x \mid x \in \mathcal{E} \wedge x \notin A\}$ is called the complement of A with respect to $\mathcal{E}$ or the absolute complement of A and is denoted by $\mathcal{E} \setminus A$. In other words, $\overline{A} = \{x \mid x \in \mathcal{E} \wedge x \notin A\}$ (Arıkan & Halıcıoğlu, 2018, p. 92).
Relative complement of a set	Let $\mathcal{E}$ be a universal set and $A, B \subseteq \mathcal{E}$. Then the relative complement of B in A is defined as $A \setminus B = \{x \mid x \in A \wedge x \notin B\}$ and similarly, the relative complement of A in B is defined as $B \setminus A = \{x \mid x \in B \wedge x \notin A\}$ (Arıkan & Halıcıoğlu, 2018, p. 93).

Table S2. *Formal Definitions of Concepts Related to the Functions Sub-learning Area*

Name of the concept	Definition
Function	Let A and B be two sets and f be a relation from A to B. If for every $a \in A$ there exists a unique $b \in B$ such that $(a, b) \in f$, then f is called a function from A to B and is denoted by $f: A \to B$. The set A is called the domain of f and the set B is called the codomain of f. If $(a, b) \in f$ for $a \in A$, then b is called the image of a under the function f and is denoted as $b = f(a)$ (Arıkan & Halıcıoğlu, 2018, p. 173).
Restriction and extension of a function	Let A, B be two sets and $A_1 \subseteq A$. (1) When a function $f: A \to B$ is given, the function $f|_{A_1}: A_1 \to B$ defined as $f|_{A_1}(a) = f(a)$ for each $a \in A_1$ is called a restriction of f to the set A_1. (2) When a function $f: A_1 \to B$ is given, a function $g: A \to B$ satisfying the condition $g(a) = f(a)$ for each $a \in A_1$ is called an extension of f to the set A (Arıkan & Halıcıoğlu, 2018, p. 177).
Injective function	Let A, B be two sets and f be a function from A to B. If $x = y$ for every $x, y \in A$ satisfying the condition $f(x) = f(y)$, then f is called an injective function (Arıkan & Halıcıoğlu, 2018, p. 179).
Surjective function	Let A, B be two sets and $f: A \to B$ be a function. If for every $b \in B$ there can be found $a \in A$ such that $b = f(a)$, then f is called a surjective function (Arıkan & Halıcıoğlu, 2018, p. 180).
Bijective	A function that is injective and surjective is called a bijection (Arıkan & Halıcıoğlu, 2018, p. 181).

Identity function	Let $\emptyset \neq A$ be a set. The function $i_A : A \to A$ defined as $i(a) = a$ for each $a \in A$ is called a identity function on the set A (Arıkan & Halıcıoğlu, 2018, p. 182).
Composition of functions	Let the sets A, B and C other than the empty set be given. When the functions $f : A \to B$ and $g : B \to C$ are given, the function $gof : A \to C$ defined as $(gof)(a) = g(f(a))$ for each $a \in A$ is called the composition of f and g (Arıkan & Halıcıoğlu, 2018, p. 184).
Inverse function	Let the function $f: A \to B$ be given for sets A and B. If f is a bijective, then the function $f^{-1} : B \to A$ is called the inverse of f (Arıkan & Halıcıoğlu, 2018, p. 189).

Supplementary Information 2:

Supporting Evidence for the Table

Table S3. *Structure of the Category of Thinking About the Relationship Between Concept and Picture (with supporting evidence)*

Subcategory	Dimension	Sub dimension
Concept: Contains the thoughts of the PMTs about the concept (e.g., Figs. 1, 2, and 3).	Reviewing the definition: Contains the PMTs' own explanations regarding the definitions of the concepts (e.g., Fig. 1).	
	Sampling: Contains examples determined by the PMTs regarding the concepts (e.g., Fig. 2).	
	Generating ideas: Includes explanations in which they state that they drew the pictures to get ideas about the concept (e.g., Fig. 3).	
Picture: Indicates the ideas at the starting point of the created picture of concept and the purpose for which the lines and colors in the picture of concept are used (e.g., Figs. 4 and 5).	Source of inspiration: Involves how PMTs identify a source of inspiration by establishing similarities between their prior knowledge and the pictures they draw when visualizing concepts (e.g., Fig. 4).	Discovering a source of inspiration: Includes the prior knowledge of PMTs that underlies the background of their pictures (e.g., Fig. 4a).
		Developing the source of inspiration: Refers to the prior knowledge of PMTs that becomes a factor in creating the picture of the concept (e.g., Fig. 4b).
	Determining representation: Refers to PMTs' attributing meanings to some colors and lines to represent the concept while creating their pictures (e.g., Fig. 5).	Color: PMTs attributed meanings to some colors while creating their pictures (e.g., Fig. 5a).
		Line: PMTs attributed meaning to some lines while creating their pictures (e.g., Fig. 5b).
Exploring a new situation: Refers to the situations in which the PMTs reach generalizations between concepts or between the concept and the picture while drawing their pictures (e.g., Figs. 6 and 7).	Concept-picture relationship: Contains generalizations about the concept and picture relationship (e.g., Fig. 6).	
	Concept-concept relationship: Includes generalizations between concepts (e.g., Fig. 7).	

Note. This table corresponds to Table 1 in the main manuscript.

Figure 1. *Thinking About the Relationship Between Concept and Picture Category, Concept Subcategory, Reviewing the Definition*

Dialogue between PMTs and Researcher (R)
G_2M_1: The symbols of the elements that sets possess do not matter at all. They could be numbers, letters, shapes-nothing makes a difference. G_2M_3: We can even use two-dimensional or three-dimensional symbols.
R: Could you explain how you made the drawing, starting with the concept of function from the beginning? G_1M: Now, a function is essentially a relation defined from one set to another. It's a special kind of relation... Set A is the domain, and set B is the codomain. For something to be a function, the first condition is that there must be no unmatched elements in the domain. Additionally, no element in the domain can have more than one image. After considering these, I...

Figure 1 shows that PMTs explained the definitions of the concepts of set and function from their own perspective.

Figure 2. *Thinking About the Relationship Between Concept and Picture Category, Concept Subcategory, Sampling*

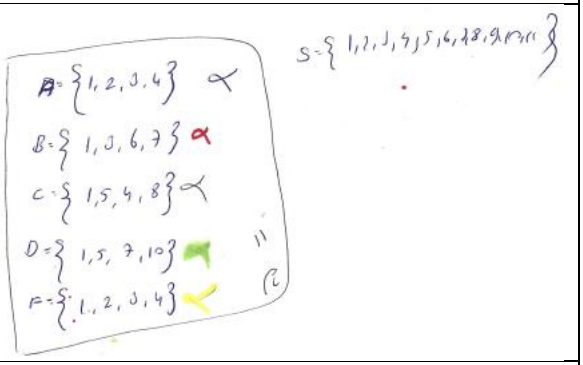
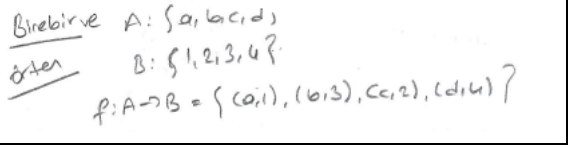
Dialogue between PMTs and R	Sample documents
G_2M_1: Then, we showed five sets for the first time, which were our own. Sets A, B, C, D, and F, and there is a universal set that includes them, which also has the number 11. R: Does the number 11 belong only to the universal set? G_2M_1: Yes.	
G_1M: Let there be a function defined from set A to set B. For this function to be bijective, it must satisfy the conditions of injectivity and surjectivity. Therefore, I selected my elements accordingly.	

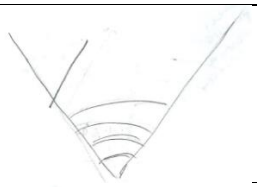
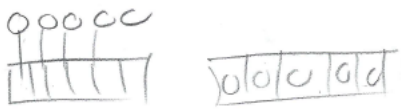
Figure 2 shows examples produced by some PMTs regarding the concepts of set and function.

Figure 3. *Thinking About the Relationship Between Concept and Picture Category, Concept Subcategory, Generating Ideas Dimension*

Dialogue between PMTs and R
R: How did you think and draw from the beginning, starting from its initial form? Feel free to explain. G_1M: When I first drew this picture... As you can see, there are squares inside squares. This also looks like a fractal example. I actually did this to understand within myself what the meaning of the set is.
R: You can explain it from the beginning. G_4M: At first, I drew two sets... A Venn diagram, to have a clearer picture in my mind. It is already related to the functions we know. I thought, let us try it on a small set and see how and what it is, and try to solve the situation through the Venn diagram.

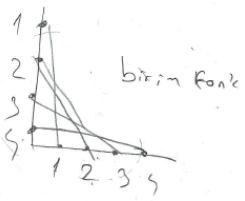
Since the pictures and explanations created by the PMTs for gaining insight into the concept are re-examined under the category of drawing a picture, only examples of the relevant explanations from the PMTs are provided in Figure 3.

Figure 4a. *Thinking About the Relationship Between Concept and Picture Category, Picture Subcategory, Source of Inspiration Dimension, Discovering a Source of Inspiration Subdimension*

Dialogue between PMTs and R	Sample documents
G_1M: Then, I made a perspective drawing here. When we made perspective drawings in art class, a road going to infinity appeared before my eyes, and I thought I would use this to represent a triangular region of sets.	
R: How did you draw the identity, surjective, and injective functions? Please explain. $G_5ÖM_1$: The first thing that came to our mind was to put stamps in a box, one stamp in each box.	

In Figure 4a, the PMTs stated that they used the *"perspective drawing"* idea when trying to draw the picture of the concept of the set, and the *"box and stamp"* idea when creating the picture of the concept of the function.

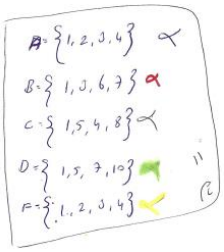
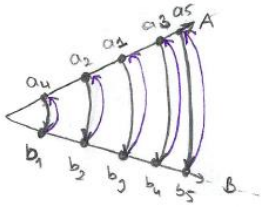
Figure 4b. *Thinking About the Relationship Between Concept and Picture Category, Picture Subcategory, Source of Inspiration Dimension, Developing the Source of Inspiration Subdimension*

Dialogue between PMTs and R	Sample documents
G_2M_1: Also, what we drew here is not a number line. It seemed pointless to us to draw a number line. Because why? The elements of the sets do not necessarily have to consist of numbers; they can also consist of the alphabet. R: So, did you draw this based on the idea of a number line? G_2M_1: Yes, there is an idea of a number line, but it is not a number line. These sets can also consist of letters of the alphabet, such as a, b, c, and d. Then, we place the letters here or the shapes in the order we like.	
G_6M: I thought I could show the identity function in this way. I thought of lines we can draw in L shapes and extend to infinity. We can extend them in this way. This is similar to the coordinate system. It came to my mind a little from there.	

In Figure 4b, the pictures of concepts drawn by the PMTs related to the concepts of sets and functions are based on the ideas of the *"number line"* and the *"coordinate system."*

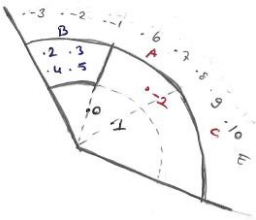
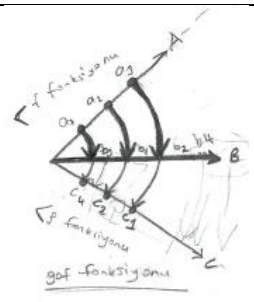
Figure 5a. *Thinking About the Relationship Between Concept and Picture Category, Picture Subcategory, Determining Representation Dimension, Color Subdimension*

Dialogue between PMTs and R	Sample documents and interpretation
G_2M_1: We assigned a color to each of the sets. Set A is represented by the color blue, set B by red, set C	

by black, set D by green, and set F by yellow. We circled those that have these elements according to their color.	PMTs determined colors for the sets in their picture.  They marked the elements of the sets on the line according to the colors they determined. 
R: In this picture, did you show the matching of the elements of the function f in black and the matching of the elements of the inverse of the function in purple? G_3M: Yes.	

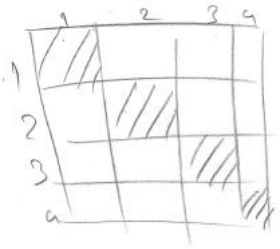
In Figure 5a, $G_2M_1M_2M_3$ represents the sets with different colors, and G_3M shows the matching of the elements of the inverse of the function with a different color.

Figure 5b. *Thinking About the Relationship Between Concept and Picture Category, Picture Subcategory, Determining Representation Dimension, Line Subdimension*

Dialogue between PMTs and R	Sample documents
G_1M: Then, when I asked how to connect them, I turned the solid line into a dotted line in order to connect them.	
R: You drew some parts thickly and some parts thinly. Why did you do that? G_3M: The thick and thin lines are for the f and g functions to be seen better. Common sets like this, for example, the codomain of one and the domain of the other. That's why I marked them with thick lines to be seen more clearly. To make it visible that the two are different functions.	

In Figure 5b, G_1M used dashed lines to show the common elements of the sets and wrote the common elements on these lines, and G_3M used thick and thin lines to make the composite function more visible.

Figure 6. *Thinking About the Relationship Between Concept and Picture Category, Exploring a New Situation Subcategory, Concept-picture Relationship Dimension*

Dialogue between PMTs and R	Sample documents
G_2M_1: We assigned a color to each of the sets. Set A is represented by the color blue, set B by red, set C by black, set D by green, and set F by yellow. We circled those with these elements according to their color. If there are color circles on the elements, it means that those elements are common elements of the sets.	
G_5M_2: Here, we drew the identity function. G_5M_1: We matched 1 with 1, 2 with 2, 3 with 3, all of them with themselves. G_5M_2: So, when the squares on the diagonal are shaded, it becomes the identity function.	

In Figure 6, $G_2M_1M_2M_3$ stated that they could see the common elements through the colors of the sets in the picture they drew, and $G_5M_1M_2$ stated that if all the shaded areas are on the diagonal, the function represented by the picture is the identity function.

Figure 7. *Thinking About the Relationship Between Concept and Picture Category, Exploring a New Situation Subcategory, Concept-concept Relationship Dimension*

Dialogue between PMTs and R	Interpretation
G_4M: But, for example, if we look at the next part with the triple, the middle triple, here we have surjectivity, but it is not injective. In other words, when we look from the reverse perspective, one element maps to two different places, so it cannot be an inverse function. For the inverse to be a function, it must also be injective.	The PMTs were defined as a function's inverse and asked to draw the picture of concept. According to the definition, for the inverse of a function to be a function, there must be a bijective (both injective and surjective). Some PMTs discovered that the inverse of such functions is not a function in their pictures of concept by removing the conditions of being injective or surjective.
G_1M: I did this only for injectivity. I randomly took two sets. The function f is not surjective from A to B, but it is injective. Now, when I take the inverse of the function from set B to set A, two elements are left empty in the domain. There are elements that do not match. The inverse still does not define a function.	

In Figure 7, G_4M and G_1M tested the conditions necessary for the inverse of a function to exist.

Supplementary Information 3:

Supporting Evidence for the Table

Table S4. *Structure of the Category of Drawing a Picture (with supporting evidence)*

Subcategory	Dimension
Familiar picture: Represents the drawings that PMTs already know about the visual representation of concepts (e.g., Fig. 8).	Venn diagram (e.g., Fig. 8)
	Function graph (e.g., Fig. 8)
	Coordinate system (e.g., Fig. 8)
Concrete picture: Contains pictures of real beings found in nature to represent the concept of PMTs (e.g., Fig.9).	
Draft picture: These are the pictures that PMTs stated that they drew to experiment while trying to draw a picture of the concepts (e.g., Fig.10).	Purposeful draft picture: These are the pictures that PMTs say they drew to try for a particular purpose (e.g., Fig.10a).

	Purposeless draft picture: These are the pictures that PMTs stated they did not have any thoughts on while drawing (e.g., Fig.10b).
Mismatched picture: Represents the pictures that differ from PMTs' explanations and other similar pictures. Such mismatched pictures emerge as a result of differences in the act of drawing (e.g., Fig. 11).	Incomplete drawing: Refers to cases in which PMTs draw their pictures incompletely compared to their explanations and other drawings (e.g., Fig. 11a).
	Excessive drawing: Refers to cases in which PMTs draw their pictures with additional or redundant elements compared to their explanations and other pictures (e.g., Fig. 11b).
Coloring: Contains all colorings of PMTs pictures of concept (e.g., Fig. 12).	
Abstract picture: Contains pictures of PMTs that do not depict real beings found in nature (e.g., Fig. 13).	Representing the concept: Some of the abstract pictures of PMTs represent only one concept (e.g., Fig. 13a).
	Representing concepts: Some of the abstract pictures of PMTs represent more than one concept (e.g., Fig. 13b).

Note. This table corresponds to Table 2 in the main manuscript.

Figure 8. *Drawing a Picture Category, Familiar Picture Subcategory, Venn Diagram, Function Graph, and Coordinate*

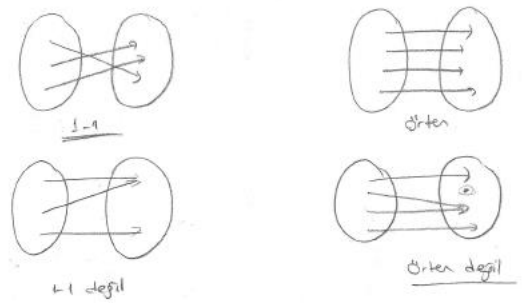
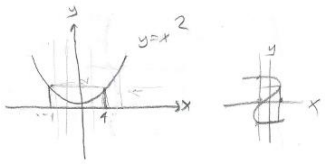
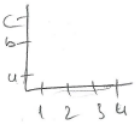
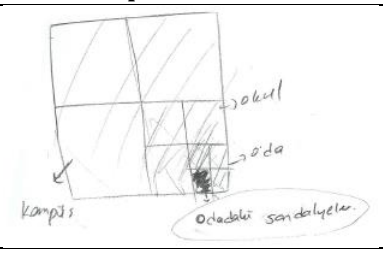
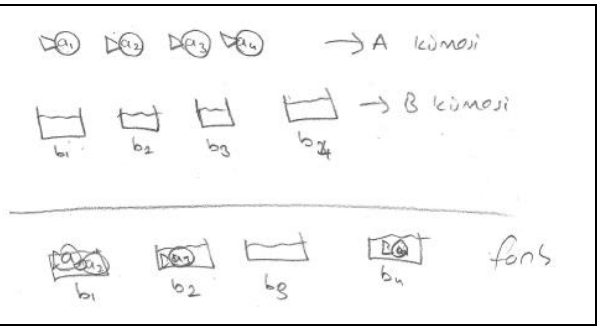
Dialogue between PMTs and R	Sample documents
R: Can you explain how you thought from the beginning and drew your picture? Did you first draw Venn diagrams? G_3M: First, I found examples of injective and surjective functions to form something in my mind and internalize the concept better.	
G_4M: We thought about how to go through the graph. How would we draw it? How would it become a function?	
G_5M_2: We actually did not do it in exactly the same way as the coordinate plane. We matched them with the help of arrows. There are two lines, and there are elements of the sets. We only took the first region of the coordinate system.	

Figure 9. *Drawing a Picture Category, Concrete Picture Subcategory*

Dialogue between PMTs and R	Sample documents
G_1M: For example, the chairs in the room form a set inside the room. The room forms a set inside the school. The school forms a set inside the campus. That was my initial idea. I thought of nested sets.	

G_5M_2: Then, we thought of it as fish and aquarium. We said that it would be more understandable. We said that every fish needs to go into an aquarium to be able to live. If a fish does not go into any aquarium, it cannot live; it dies; in other words, it does not function. G_5M_1: Then, the function is not injective. There may be an empty aquarium, but no empty fish should exist.	

In Figure 9, G_1M associated the set concept with the objects "campus, school, room, and chair", and $G_5M_1M_2$ tried to represent the function with the "aquarium and fish".

Figure 10a. *Drawing a Picture Category, Draft Picture Subcategory, Purposeful Draft Picture Dimension*

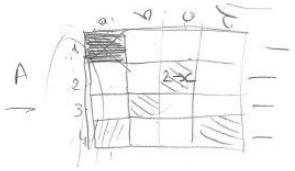
Dialogue between PMTs and R	Sample documents
G_5M_1: I first drew a square as a draft. We divided the square into squares again. Then we said, let us take this part of the square as the domain, set A; that is what we thought then. This became set B, the codomain. Then, we said that if we color inside these small squares, we can show that 1 goes to a.	

Figure 10b. *Drawing a Picture Category, Draft Picture Subcategory, Purposeless Draft Picture Dimension*

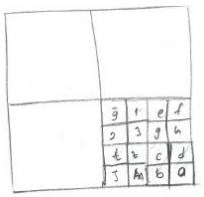
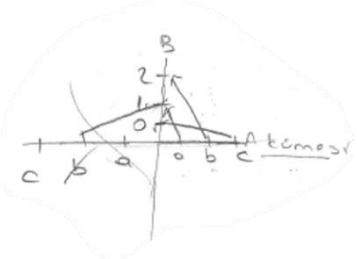
Dialogue between PMTs and R	Sample documents
R: What is the drawing right next to this one? Did you continue drawing there? G_1M: I just tried adding letters there; I had no thoughts about it.	

Figure 11a. *Drawing a Picture Category, Mismatched Picture Subcategory, Incomplete Drawing Dimension*

Dialogue between PMTs and R	Sample documents and interpretation
R: In this draft picture, there is a line segment and directed arrows. In your other pictures, there are only directed arrows. G_5M_1: Then we placed them here; we forgot to place them here.	$G_5M_1M_2$ represented the correspondence between the elements of the function in the draft picture using both a line segment on the left side (matching b with 1) and directed arrows on the right side (matching a, b, and c with 1, 2, and 0, respectively). 

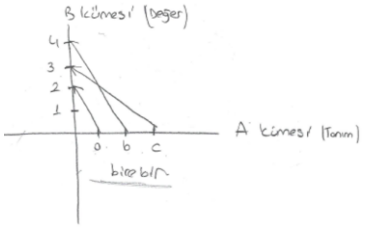
	In the other pictures, $G_5M_1M_2$ represented the correspondence only by using directed arrows. When the researcher asked about the reason, they stated that they were unaware of it. 

Figure 11b. *Drawing* a *Picture Category, Mismatched Picture Subcategory, Excessive Drawing Dimension*

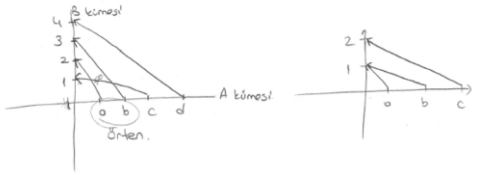
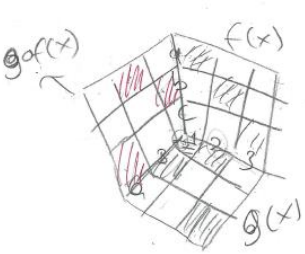
Dialogue between PMTs and R	Sample documents and interpretation
G_5M_1: Then we said that this drawing should not look like it shows four regions, but like it shows a single region, as students might get confused. R: You thought so. However, it seems like you drew those areas in your other drawings. G_5M_1: Teacher, our hand slipped a little. However, what we wanted to draw is this, it is visible here. Actually, we tried to draw and show it first. Then, we tried to minimize it as a design. We decided to make it like the shape we drew in that corner.	While explaining their picture of the function, $G_5M_1M_2$ stated that they were thinking of the first region of the coordinate system. The researcher asked whether the other regions of the coordinate system appeared to be drawn in their pictures (below).  $G_5M_1M_2$ stated that the primary drawings they draw due to their manual habit are like the first region of the coordinate system (below). 

Figure 12. *Drawing a Picture Category, Coloring Subcategory*

Sample documents and interpretation	
$G_2M_1M_2M_3$ determined the color blue for set A, red for set B, black for set C, green for set D, and yellow for set F. They placed all the elements on a line they drew and drew circles, or put dots according to the colors of the sets to which the elements belonged.	$G_5M_1M_2$ showed the matching of elements in the composition of functions in red. 

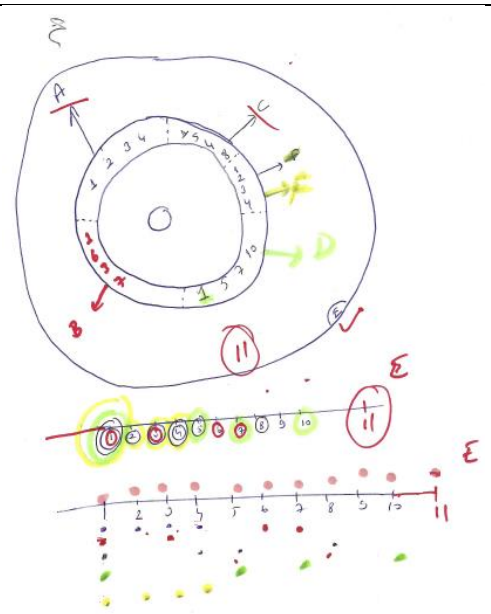
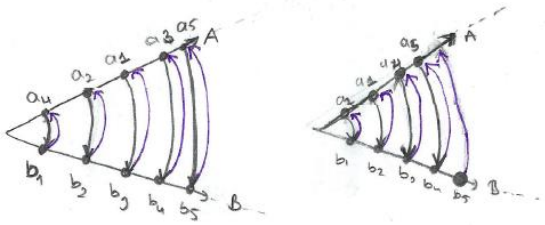
	G_3M showed the pairing of elements of the inverse function and non-invertible function in purple. 

Figure 13a. *Drawing* a *Picture Category, Abstract Picture Subcategory, Representing the Concept*

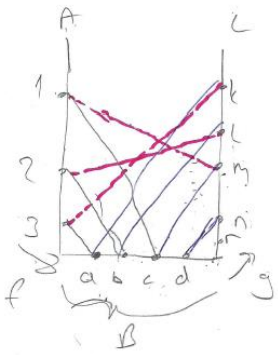
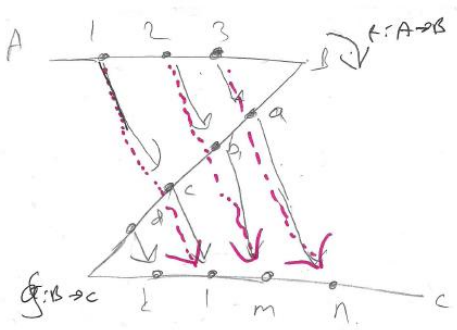
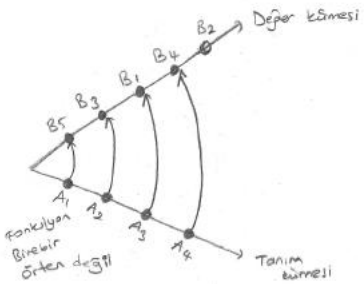
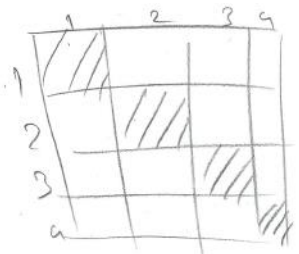
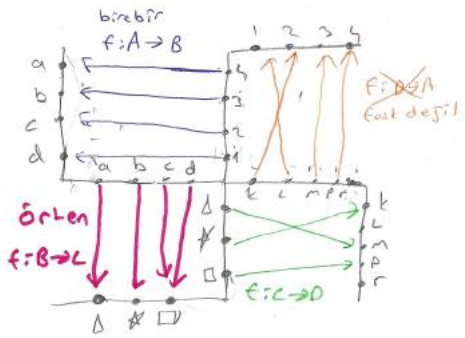
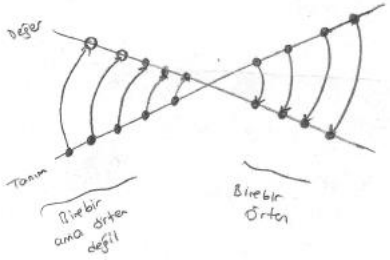
Sample documents and interpretation	
G_6M drew two different pictures of the concept of the composition of functions.  	
G_3M drew an injective function but not a surjective one, indicating that an empty dot represents the element (B_2) that does not match in the codomain. 	$G_5M_1M_2$ drew a picture of the concept of the identity function. 

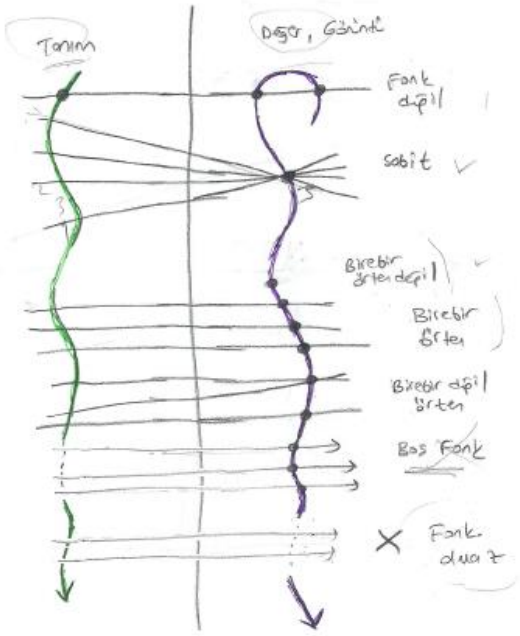
Figure 13b. *Drawing* a Picture Category, Abstract Picture Subcategory, Representing Concepts Dimension

Sample documents and interpretation	
G_6M drew a picture of the concept of functions. The picture of the concept shows not a function, function, injective function, and surjective function.	G_3M tried to represent an injective function but not a surjective one on one side of the drawing, and an injective and surjective function on the other. G_3M also drew matching lines through the empty dots but indicated that the empty dots were the non-matching elements.





G_4M drew a picture of the concept of function types. G_4M tried to represent not a function, a constant function, an injective function but not a surjective one, an injective and surjective function, a surjective function but not an injective one, and the empty function in the picture of the concept.



Supplementary Information 4:

Supporting Evidence for the Table

Table 5S. *Structure of the Category of Evaluation of the Picture (with supporting evidence)*

Subcategory	Dimension	Subdimension
Evaluation of the picture itself: Contains the situational assessments made regarding the picture itself (e.g., Fig. 14 and 15).	Representation: Contains statements that PMTs consider the picture sufficient or insufficient regarding representation (e.g., Fig. 14).	Insufficient (e.g., Fig. 14a)
		Sufficient (e.g., Fig. 14b)
	Design: Contains statements that PMTs consider the picture sufficient or insufficient regarding design (e.g., Fig. 15).	Insufficient (e.g., Fig. 15a)
		Sufficient (e.g., Fig. 15b)
Evaluation by comparing with another picture: Refers to comparing the pictures drawn by the PMTs with each other (e.g., Fig. 16).	Design: Includes statements in which PMTs evaluate a picture as successful or unsuccessful regarding design compared to another picture they drew (e.g., Fig. 16).	Unsuccessful (e.g., Fig. 16)
		Successful (e.g., Fig. 16)

Note. This table corresponds to Table 3 in the main manuscript.

Figure 14a. *Evaluation of the Picture Category, Evaluation of the Picture Itself Subcategory, Representation Dimension, Insufficient Subdimension*

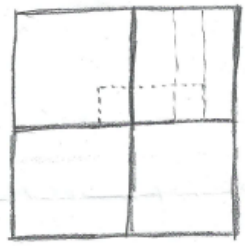
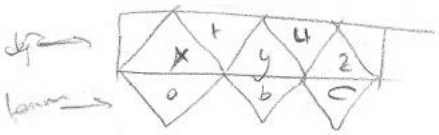
Dialogue between PMTs and R	Sample documents
G_3M: Then I said, Can I go from the square? Can I match it from the square like this? I could not get it out of there.	
R: What did you draw next? G_4M: Sir, this… This is the domain; this is the codomain. I tried to create something like a pattern that is different from similar shapes, but. As I said, this is limited again, for example. We can only show it with shapes that are similar to each other. R: Did you think there would be no representation for a general function? G_4M: Yes, that did not work either.	

Figure 14b. *Evaluation of the Picture Category, Evaluation of the Picture Itself Subcategory, Representation Dimension, Sufficient Subdimension*

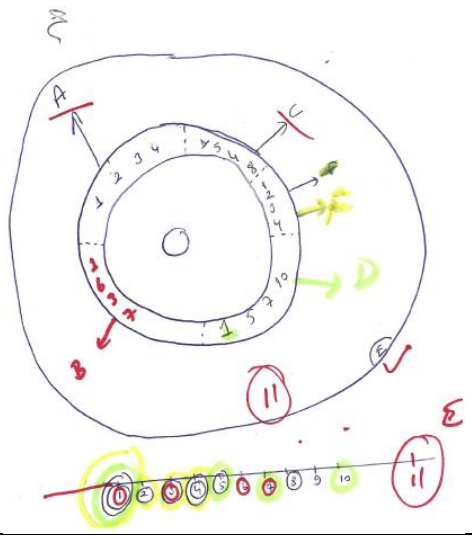
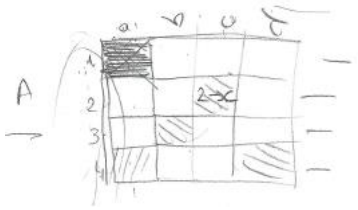
Dialogue between PMTs and R	Sample documents
G_2M_1: At first, we thought like this: when we circled each element with circles, we could clearly see the intersections of the sets having those colors or whether the element with a single color was in other sets.	
G_5M_1: I first drew a square as a draft. We divided the square into squares again. Then we said, let us take this part of the square as the domain, set A; that's what we thought at that moment. This became set B, the codomain. Then we said that if we color inside these little squares, we can show that 1 goes to a. This seemed logical.	

Figure 15a. *Evaluation of the Picture Category, Evaluation of the Picture Itself Subcategory, Design Dimension, Insufficient Subdimension*

Dialogue between PMTs and R	Sample documents

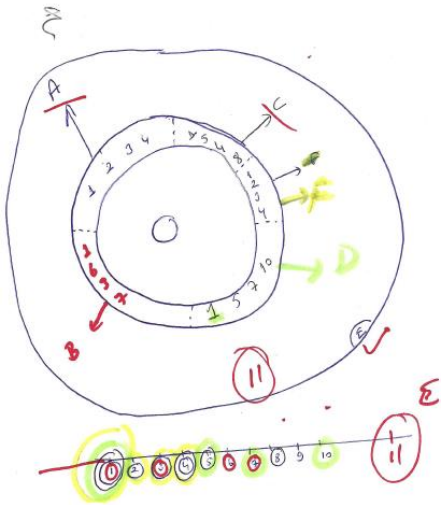
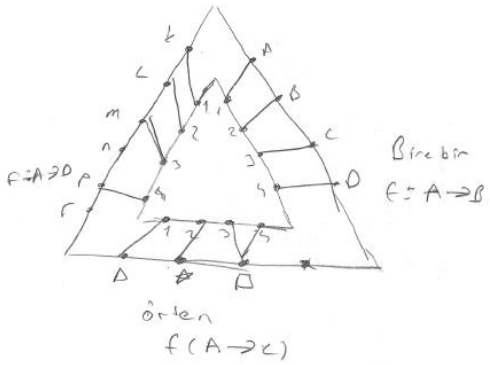
G_2M_3: We had drawn a circle, but it was more complicated. So, we considered it more appropriate to put a dot. G_2M_1: Yes, it overlaps here when too many lines are drawn... R: Then explain why you altered it. G_2M_2: When it becomes a circle, its radius grows and looks uneven. We draw one to one, two to another, three to another, four to another. R: Your first drawing was in the form of concentric circles. Did you put dots under the shape because you thought it would not look right as it grew? G_2M_1: Yes, when we look at it this way…	
G_6M: At first, this shape was right in my head, but then I thought... I thought about drawing a more open shape that is very closed.	

Figure 15b. *Evaluation of the Picture Category, Evaluation of the Picture Itself Subcategory, Design Dimension, Sufficient Subdimension*

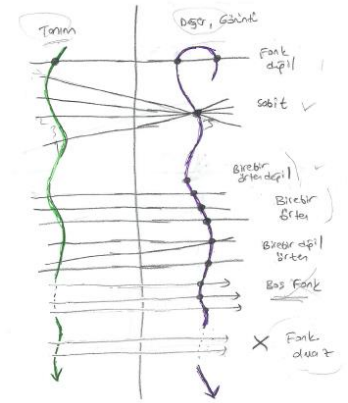
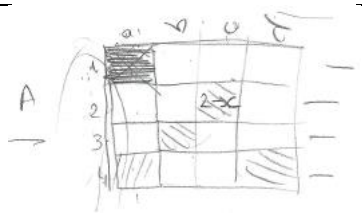
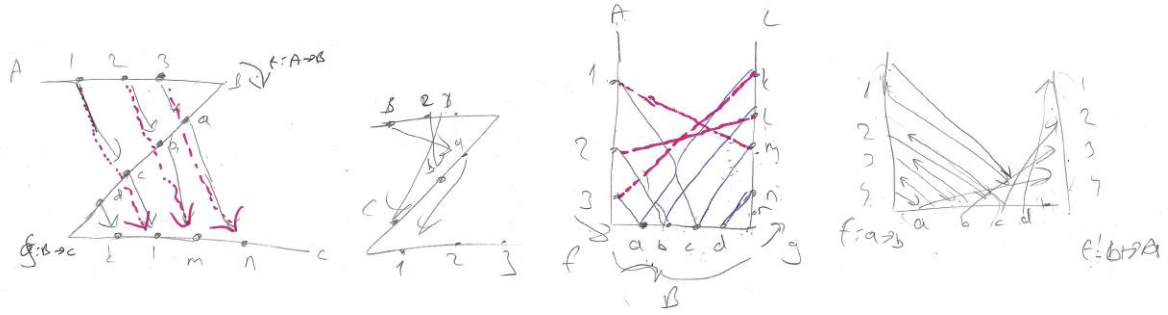
Dialogue between PMTs and R	Sample documents
R: Do the curved lines represent a set, and the straight lines represent a match? G_4M: Yes. It will not get confusing, and it will be more apparent.	
R: Did you see that it could be possible in that draft? G_5M_1: Yes. Also, well… How should I put it? When you look at it, it can be understood. It is not something very complicated. It is immediately apparent.	

Figure 16. *Evaluation of the Picture Category, Evaluation by Comparing with Another Picture, Design Dimension, Successful and Unsuccessful Subdimensions*

Interpretation, sample documents, and dialogue between pupils and R

G_6M represented the composition of functions (first and third picture from the left) and the inverse function (second and fourth picture from the left) in the form of "z" and "u." When he compared the two shapes, he stated that he considered the "z" shape more successful than the "u" shape in terms of design. Therefore, he considered the "u" shape less successful in terms of design.  G_6M: I thought of the u and z shapes similarly in the inverse function. However, the z shape looks better in the composition of functions and inverse functions.

Supplementary Information 5:

Supporting Evidence for the Table

Table 6S. *Structure of the Category of Altering the Picture (with supporting evidence)*

Subcategory	Dimension
Representation: Includes the alterations PMTs made regarding the representation of concepts in their pictures (e.g., Fig. 17).	Partially: Includes the partial alterations PMTs made regarding the representation of concepts in their pictures (e.g., Fig. 17a).
	Completely: Includes the complete alterations PMTs made regarding the representation of concepts in their pictures (e.g., Fig. 17b).
Design: Includes the alterations PMTs made regarding design in their pictures (e.g., Fig. 18).	Partially: Includes the partial alterations PMTs made regarding the design of concepts in their pictures (e.g., Fig. 18a).
	Completely: Includes the complete alterations PMTs made regarding the design of concepts in their pictures (e.g., Fig. 18b).
Size: Includes the alterations PMTs made regarding size in their pictures (e.g., Fig. 19).	Shrinking: Includes PMTs shrinking the size of their pictures (No example was observed).
	Enlargement: Includes PMTs enlarging the size of their pictures (e.g., Fig. 19).

Note. This table corresponds to Table 4 in the main manuscript.

Figure 17a. *Altering the Picture Category, Representation Subcategory, Partially Dimension*

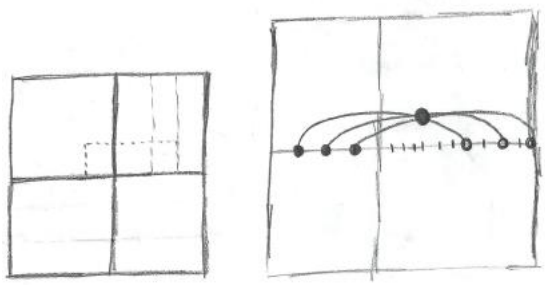
Dialogue between PMTs and R	Sample documents and interpretation
G_3M: Then I said, can I go from the square? Can I match it like this from the square? I could not get it out of there. Then I started to take dots on the square again, and this time, I thought of the other side as a codomain and matched it.	G_3M tried to represent the function by drawing a square (the picture on the left). She thought the shape she drew could not represent the function and altered some of its shape (the picture on the right). 

Figure 17b. *Altering the Picture Category, Representation Subcategory, Completely Dimension*

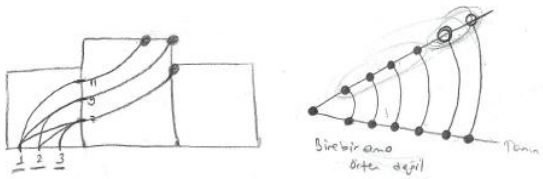
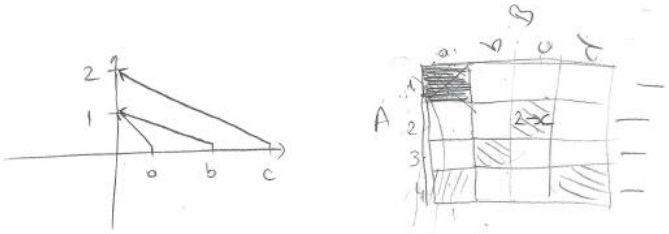
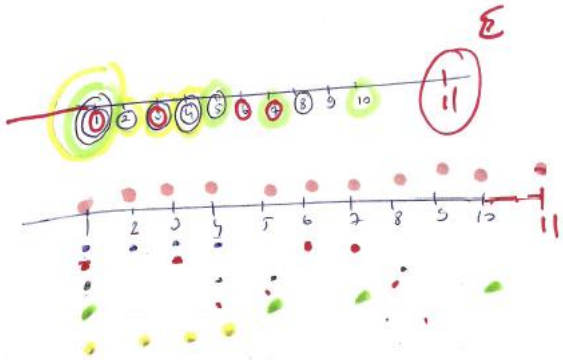
Dialogue between PMTs and R	Sample documents and interpretation
G_3M: Then, I decided to combine three squares or so. R: Why did you combine the three squares? G_3M: I wanted to show the surjectivity on this side and the injectivity on this side. However, the lines would get mixed up there, too, so I gave up on that too. Then I decided to do something like a bow, something angled.	G_3M completely abandoned the drawings with squares (the picture on the left) and began experimenting with new drawings inspired by the ideas of angles and arcs (the picture on the right). 
R: How did you think then? G_5M_1: Then, we gave up on it because it was a representation similar to a coordinate system. So, we gave up on it. G_5M_2: We decided to think of something different since it was similar to a coordinate system. G_5M_1: Then we moved on to here. I first drew a square as a draft. We divided the square into squares again. Then we said, let us take this part of the square as the domain, set A; that is what we thought at that moment. This became set B, the codomain. Then we said if we color inside these little squares, we can show that 1 goes to a. This seemed logical to us.	$G_5M_1M_2$ tried to represent the function using the coordinate system (the picture on the left). They thought their drawings were not notably different from the coordinate system and tried to represent the function by drawing a square (the picture on the right). 

Figure 18a. *Altering the Picture Category, Design Subcategory, Partially Dimension*

Dialogue between PMTs and R	Sample documents and interpretation
G_2M_3: In the second drawing, we put dots underneath. R: How? G_2M_3: We had drawn a circle, but it was more complicated. So, we decided to put a dot. G_2M_1: Yes, it overlaps here when too many lines are drawn… R: Then explain why you altered it. G_2M_2: When it becomes a circle, its radius grows and looks uneven. We draw one to one, two to another, three to another, four to another… R: Your first drawing was in the form of concentric circles. Did you put dots under the shape because you thought it would not look right as it grew? G_2M_1: Yes, when we look at it this way …	$G_2M_1M_2M_3$ started to show the elements of the sets with colored dots (the picture on the bottom) instead of showing them with nested colored circles (the picture on the top). 
R: Then, I guess you drew a shape similar to the one you drew. G_3M: It is the same, actually. I just slightly turned it sideways since this is my main shape. R: Why did you draw the same thing again? G_3M: My aim was just to show the composition of functions. However, then I wanted it to look like this. Because I am not	G_3M drew a picture representing the composition of functions (the picture on the left). She thought this picture was not consistent in design with her other pictures and therefore redrew the same picture by changing its orientation to make it resemble the others (the picture on the right).

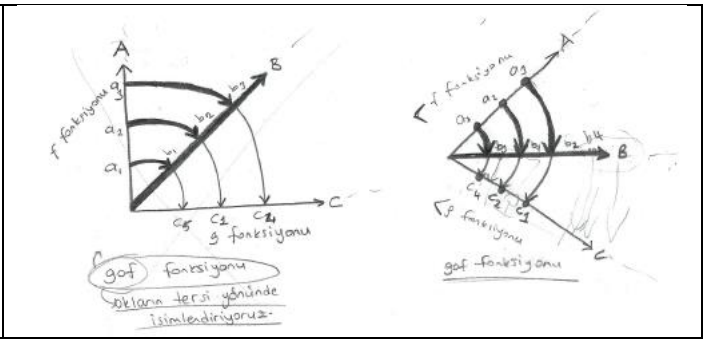
going from this to that. I just turned it sideways so it would be the same shape. I did not add anything else.	

Figure 18b. *Altering the Picture Category, Design Subcategory, Completely Dimension*

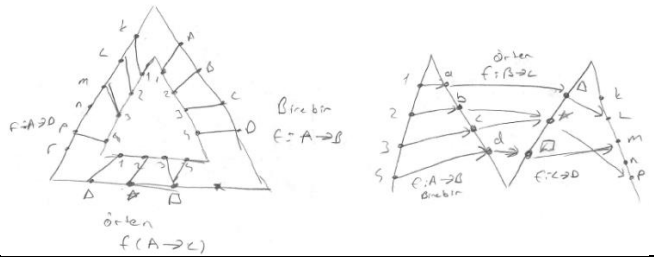
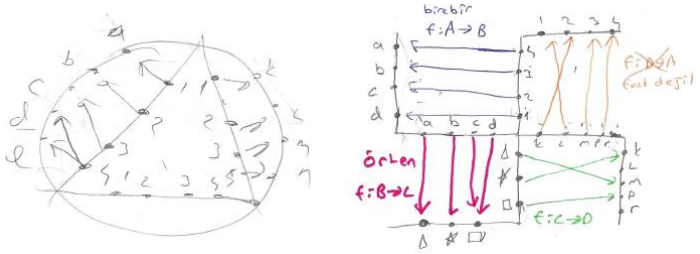
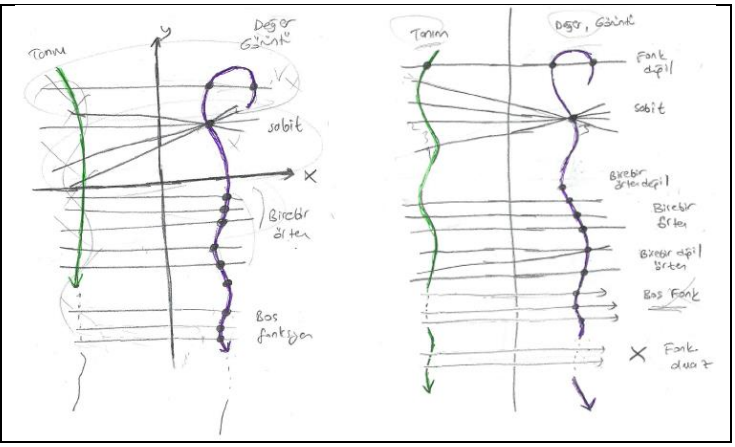
Dialogue between PMTs and R	Sample documents and interpretation
G_6M: I drew a triangle on the inside, and the triangle had the same domain on each side. I drew another triangle on the outside with different codomains on the sides. At first, I thought that we could show the injectivity, surjectivity, etc. of all kinds of functions here. This shape suited me at first, but then I thought… I thought about how I could draw a more open shape that is very closed. Then, I thought of showing the domain and codomain of the function with line segments instead of placing them on the sides of the triangle.	G_6M drew a picture in which he placed the domain and codomain of functions on the sides of a triangle (the picture on the left). He did not have any problems in representing the functions in this picture. However, he considered his picture a closed shape and found it insufficient in terms of design. He decided to design his picture differently and drew another picture in which he represented the domain and codomain with line segments (the picture on the right). 
G_6M: I thought of drawing a triangle inside the circle. Then, when I looked at it, I thought it was logically the same as the triangle inside the circle that I had drawn at first. Like that, this one became a closed shape. I gave up on this shape. Then I considered whether I could design something that shows them all separately. That is, show each property separately in each corner. I drew this shape.	G_6M tried to show the concepts related to function by drawing a triangle inside a circle (the picture on the left). He thought that this picture was not a precise shape regarding design and altered his design completely. He drew a picture representing not a function, a function, an injective function, and a surjective function by combining the line pieces (the picture on the right). 

Figure 19. *Altering the Picture Category, Size Subcategory, Enlargement Dimension*

Dialogue between PMTs and R	Sample documents and interpretation
R: Then, did you draw the small shape larger? G_4M: Yes, I realized I can show many properties if I extend it like this.	In order to obtain a picture that represents more properties of the functions, G_4M redrew the picture on the left by enlarging it and obtained the picture on the right below.

	Tanım Değer Görüntü sabit Birebir örten Bas fonksyon Tanım Değer, Görüntü Fonk değil sabit Birebir örten değil Birebir Örten Birebir değil örten Bas Fonk Fonk. değil