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A Comparison of Three Methods Based on the Bernstein Operational Matrix for Solving Differential and Integral Equations

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Highlights

- A novel method combining Bernstein polynomials and Aboodh integral transformation is introduced.
- For the combined method, the theoretical framework and implementation strategy are explained.
- Comparative analysis with established methods for differential and integral equations is provided.

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Abstract

This article investigates combining traditional methods for solving differential equations with Aboodh integral transformation and the Bernstein method. This approach offers a new perspective in solving various differential and integral equations. The solution approach obtained from the combination of Bernstein polynomials and integral transformations stands out in terms of theoretical and practical implications. This method can be applied to various examples of differential equations, and its practical applicability can be evaluated through studies conducted using programming languages. The error between the exact solution and the proposed method's approximate solution has been calculated. We compare the numerical results of the Aboodh-Bernstein, Laplace-Bernstein, and Sumudu-Bernstein methods.

1. INTRODUCTION

Integral transforms have played a crucial role in applied mathematics, mathematical physics and engineering for nearly two centuries, serving as fundamental tools for solving a wide range of complex problems. The idea behind integral transforms is to convert differential and integral equations into simpler algebraic forms that are easier to analyze and solve. Historically, the development of integral transforms dates back to the early 19th century, when Joseph Fourier's pioneering work *La Théorie Analytique de la Chaleur* (1822) introduced the Fourier transform. Around the same time, Pierre-Simon Laplace made significant strides in the development of the Laplace transform, which he initially applied to probability theory and later to the analysis of differential equations [1,2].

Today, the Laplace transform is a fundamental technique used across various scientific and engineering disciplines to solve both linear and nonlinear problems. While the forward Laplace transform is relatively straightforward, the inverse Laplace transform poses a more significant challenge, often requiring advanced numerical techniques. Numerous methods have been developed to invert Laplace transforms, including the use of Fourier series, eigenfunction expansions and the Talbot algorithm. These methods have made it possible to apply the Laplace transform to a wider range of complex problems [3,4].

Bernstein polynomials and their associated operational matrices are widely used in solving various differential, integral equations. It has been previously reported that the Bernstein operational matrix of integration is employed to compute the numerical inverse Laplace transform for certain functions. A method has been proposed for obtaining the numerical inverse Laplace transform using the Bernstein operational matrix of integration, applicable to linear differential, integral equations and nonlinear integral

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equations. The proposed approach expresses the solution of the equation in terms of a truncated Bernstein polynomial series and then the numerical inverse Laplace transform is obtained using the operational matrix of integration [5,6].

The combination of the Laplace transform with the Adomian Decomposition Method (ADM) represents an important development in the field of differential equations. The Adomian method, introduced by George Adomian, is a powerful tool for solving nonlinear differential equations. When combined with the Laplace transform, the Adomian Decomposition Method (LADM) becomes even more effective in simplifying and solving nonlinear and linear differential equations [7,8].

The Sumudu transform is another integral transform that has gained attention in recent years for its simplicity and effectiveness in solving differential equations. Introduced by Watugala in 1993, the Sumudu transform is conceptually similar to the Laplace transform but offers certain advantages in specific applications [9,10].

Recent research has explored the combination of the Sumudu transform with Bernstein polynomials to develop new numerical methods for solving differential equations. Bernstein polynomials are a set of orthogonal polynomials that are often used to approximate functions. Their application, in conjunction with integral transforms like the Sumudu transform, provides a powerful method for solving complex differential, integral and integro-differential equations. In this combined approach, the Sumudu transform is first applied to the differential equation to simplify the problem into an algebraic form. Then, Bernstein polynomials are used to approximate the unknown function. This method allows for accurate numerical solutions while maintaining the simplicity of the original equation. It has been particularly useful for solving nonlinear equations and has shown great promise in areas such as engineering and physics, where precise and efficient numerical methods are required [11].

The Aboodh transform, introduced by Khalid Aboodh in 2013, is a relatively new integral transform that has quickly gained recognition for its ability to solve ordinary and partial differential equations. The Aboodh transform is derived from the standard Fourier integral equation and shares similarities with both the Laplace and Fourier transforms. One of the main advantages of the Aboodh transform is its simplicity, which allows for straightforward mathematical operations. This makes it a useful tool for solving complex differential equations in fields such as engineering, physics, and applied mathematics. The Aboodh transform has been successfully applied to a wide range of problems, including linear and nonlinear differential equations and it continues to be an area of active research in the mathematical community [12-16].

The Aboodh-Bernstein method is an innovative approach that combines the Aboodh transform with Bernstein polynomials to solve differential equations numerically. In this method, the Aboodh transform is first applied to convert the differential equation into an algebraic form. Then, Bernstein polynomials are used to approximate the solution, allowing for a numerical solution to be obtained. This approach has been particularly effective in solving linear and nonlinear differential equations, providing accurate results with minimal computational effort. The Aboodh-Bernstein method has been successfully applied to a variety of problems in engineering and physics, offering a powerful new tool for researchers. This method allows for the efficient numerical solution of complex differential equations, making it a valuable addition to the toolbox of mathematicians and engineers working in applied fields [17,18].

In this paper, the main objective is to provide a comparative analysis of various integral transformation methods and their applications, with a specific focus on the Aboodh-Bernstein method. Integral transforms, such as the Laplace, Fourier, and Sumudu transforms, have been fundamental tools in solving differential equations, simplifying complex problems in applied mathematics, physics and engineering. Over the years, these methods have been refined and extended to address a broader range of challenges, from linear and nonlinear differential equations to integral equations.

Among the many integral transforms, the Aboodh transform, introduced in 2013, has gained prominence due to its simplicity and effectiveness. In combination with Bernstein polynomials, this novel approach,

referred to as the Aboodh-Bernstein method, has established itself as a robust and efficient numerical framework for the solution of differential equations. By transforming differential equations into algebraic equations and using Bernstein polynomials to approximate solutions, this method achieves high accuracy with minimal computational effort.

This paper aims to compare the Aboodh-Bernstein method with other established techniques, such as the Laplace transform and Sumudu transform, and their combinations with numerical methods, highlighting their strengths and limitations. Through this comparative analysis, we seek to evaluate the efficiency, accuracy, and applicability of the Aboodh-Bernstein method in solving complex problems in engineering, physics and applied mathematics. The rest of this paper is arranged as follows: in section 2, we introduced some fundamental definitions and preliminary concepts related to the Aboodh transform. Section 3 provided a concise explanation of the Aboodh transform iterative method for solving differential and integral equations, accompanied by a few examples to demonstrate its effectiveness. Finally, the conclusions were presented in section 4.

2. PRELIMINARIES

2.1. Aboodh Transformation

Definition 1. Aboodh transformation

$A = \{f(t): \exists\, M, k_1, k_2 > 0, |f(t)| < Me^{-vt}\}$
to be

$$A[f(t)] = A(v) = \frac{1}{v}\int_0^{\infty} e^{-vt} f(t)dt \;\; t \geq 0\,, k_1 \leq v \leq k_2$$

it is defined by the integral equation [13].

Theorem 1.
Existence of Aboodh transformation

$f, [0, \infty)$ piecewise continuous and $t > t_0$ for α- Let be a function of exponential order. In this case, $A[f(t)]$ exists for $v > \alpha$ [13-15].

The inverse Aboodh transform, analogous to the Aboodh transform, can also be defined using an integral. However, computing this integral requires the use of complex variables. Therefore, we will provide the definition of the inverse Aboodh transform parallel to the Aboodh transform.

$F(v) = \frac{2}{v^4}$.

The inverse Aboodh transform of a function is defined as follows

$$A^{-1}\left\{\frac{2}{v^4}\right\} = A^{-1}\left\{\frac{2!}{v^4}\right\} = t^2.$$

Example 1
Aboodh transformation of e^{at} function

$$A[e^{at}] = \frac{1}{v}\int_0^{\infty} e^{at}e^{-vt}\,dt = \frac{1}{v}\int_0^{\infty} e^{t(a-v)}dt = \frac{1e^{-t(v-a)}}{v(a-v)}\bigg|_0^{\infty} = 0 - \frac{1}{v(a-v)} = \frac{1}{v(v-a)}$$

Example 2
Aboodh transformation of t function

$$A[t] = \frac{1}{v}\int_0^{\infty} f(t)e^{-vt}\,dt \; = \frac{1}{v}\int_0^{\infty} te^{-vt}dt = \frac{1}{v^3}$$

Table 1 presents the Aboodh transformations of some functions.

Table 1. *Aboodh transformations of some functions*

$A[f(t)]$	$A(v)$	$A[f(t)]$	$A(v)$
$A[1]$	$\frac{1}{v^2}$	$A[tsinat]$	$\frac{2av}{v(v^2+a^2)^2}$
$A[t^n]$	$\frac{n!}{v^{n+2}}$	$A[tcosat]$	$\frac{v^2-a^2}{(v^2+a^2)^2}$
$A[e^{at}]$	$\frac{1}{v^2-av}$	$A[e^{-at}sinbt]$	$\frac{b}{v[(v+a)^2+b^2]}$
$A[sin(at)]$	$\frac{a}{v(v^2+a^2)}$	$A[e^{-at}cosbt]$	$\frac{v+a}{v[(v+a)^2+b^2]}$
$A[cos(at)]$	$\frac{1}{v^2+a^2}$	$A[e^{at}sinbt]$	$\frac{b}{v[(s-a)^2+b^2]}$
$A[t^n e^{-at}]$	$\frac{n!}{v(v+a)^{n+1}}$	$A\left[\frac{1}{\sqrt{t}}\right]$	$\frac{1}{v}\sqrt{\frac{\pi}{v}}$

2.2. Bernstein Polynomials

Bernstein polynomials are special polynomials used to approximately represent the behavior of a function over a given interval. These polynomials are based on the assumption that a function on a given interval is continuous and has maximum and minimum values in a given interval [19,20].

Definition 2. Mathematically, the Bernstein polynomial $B_n(f,x)$ for the function f defined on the interval $[0,1]$ is defined as:

$$B_n(f,x) = f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}\,.$$

Here:

- n indicates the degree of the Bernstein polynomial.
- $f(x)$ is the function that is intended to be approximately represented.
- $\binom{n}{k}$ represents the combination of n and k.

Bernstein polynomials are often used to approximately represent the behavior of a function over a certain range. This approach is widely used in solving mathematical problems, especially differential equations [21-27].

2.3. Error Determination and Convergence Review

We have examined the error assessment of the proposed method. The error function $e_k(t)$ of the truncated Bernstein expansion $f_k(t)$ is defined as

$$e_k(t) = f(t) - f_k(t).$$

From the following theorem, the maximum error limit can be approximated:

Theorem 2. Let $f(t)$ be the function defined on the interval $[0,1]$. Then, the upper limit for the errors in the truncated Bernstein expansion can be approximated.

Proof Assume that $f_k(t)$ and $f_{k+1}(t)$ are two successive approximate solutions of the given differential equation. If the error sequence is either increasing or decreasing, we can apply the triangle inequality.

$$\left|\left\|f(t)-f_{k_1}(t)\right\|_\infty-\left\|f(t)-f_{k_1}(t)\right\|_\infty\right|\le\left\|f_{k_1}(t)-f_{k_2}(t)\right\|_\infty,$$

to the approximate solutions at k and $k+1$, leading us to obtain

$$\left|\left\|f(t)-f_{k_1}(t)\right\|_\infty-\left\|f(t)-f_{k_2}(t)\right\|_\infty\right|=\beta\|f(t)-f_m(t)\|_\infty\le\|f_{k+1}(t)-f_k(t)\|_\infty,$$

$0\le\beta<1,$
where

$$\|f(t)-f_m(t)\|_\infty=max\{\|f(t)-f_{k1}(t)\|_\infty,\|f(t)-f_{k+1}(t)\|_\infty\}$$

and $\beta=\frac{\|f_{k+1}(t)-f_k(t)\|_\infty}{\|f(t)-f_m(t)\|_\infty}$.

Thus, it can be concluded that the error can be restricted from above. One of the absolute errors $e_k(t)$ or $e_{k+1}(t)$ is limited by $\|f_{k+1}(t)-f_k(t)\|_\infty$ if the error sequence is monotonic. The calculable error limit is defined for $0\le\beta<1$. However, the solution diverges when the Bernstein series diverges for $\beta>1$.

3. THE PROPOSED ABOODH-BERNSTEIN METHOD AND ITS IMPLEMENTATION

In this section, we propose a novel hybrid method that combines the Aboodh integral transformation and Bernstein polynomials to solve linear and nonlinear differential and integral equations. While the Aboodh transform simplifies the original problem into an algebraic form, the Bernstein polynomials are used to approximate the solution function effectively. The synergy of these two approaches results in a robust numerical method with high accuracy and low computational cost. Below, we outline the theoretical background and implementation steps of the proposed method.

3.1. Description of Technique Method

We describe linear or nonlinear differential equation

$$a_n(t)\frac{d^ny}{dt^n}+a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}}+\cdots+a_1(t)\frac{dy}{dt}+a_0(t)y=g(t)$$

and integral equation

$$f(t)+\alpha\int_0^t f(x)dx=\int_0^t u(x)dx$$

apply the Aboodh transform as

$$A\left[a_n(t)\frac{d^ny}{dt^n}\right]+A\left[a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}}\right]+\cdots+A\left[a_1(t)\frac{dy}{dt}\right]+A[a_0(t)y]=A[g(t)]$$

$A[f(t)]+A[\alpha\int_0^t f(x)dx]=A[\int_0^t u(x)dx],$

we obtain the solution $F(v)$. Then, we will apply the values of the matrix M_{k+1} in place of s like $F(M_{k+1})$.

in the next step

$d = [d_0, d_1, d_2, \dots, d_k]^T$

$d_i = \int_0^t B_{i,k}(t)dt \quad i = [0,1, \dots, k]\,, \quad t = [0,1].$

In the final step, we find the values of the matrix C^T using the formula:

$C^T = d^T F(M_{k+1})$

and solve for the approximate solution

$f_k(t) = C^T B(t).$

As a result, for $k = 6$ we solve the approximate solution in the interval $t = [0,1]$ using the Bernstein polynomial.

Error function , $e_k(t) = f(t) - f_k(t)$
Relative error, $Error = \frac{|f(t)-f_k(t)|}{|f(t)|}$

3.2. Computational Experiments

The matrix M_{k+1} represents the integral operator matrix of Bernstein polynomials, which are part of the orthonormal Bernstein polynomials used to solve differential, integral, and integro-differential equations. Below is the derivation process for the matrix M_{k+1} :
Operational Matrix of Bernstein Polynomials
Bernstein polynomials in the interval $[0,1]$ are defined as:

$b_{i,k}(t) = \binom{k}{i} t^i (1-t)^{k-i} \quad , \quad i = 0,1, \dots, k.$

By applying the Gram-Schmidt orthonormalization process, these polynomials are transformed into an orthonormal form:

$B(t) = \left[B_{0,k}(t), B_{1,k}(t), \dots, B_{k,k}(t)\ \right]^T.$

Operational Integration Matrix:

The operational integration matrix for orthonormal Bernstein polynomials is derived based on the integration property:

$$\int_0^t B(x)dx = M_{k+1}\, B(t).$$

Here, M_{k+1} represents a fixed matrix of size $(k+1) \times (k+1)$.

Derivation Process:
This matrix is derived by expressing the integral in terms of the Bernstein basis. For each component, the following definition is used:

$$\int_0^t B_{i,k}(x)dx = \sum_{j=0}^{k} a_{i,j}\, B_{j,k}(t)$$

where $a_{i,j}$ denotes the integration coefficients.
When written in matrix form, the integration matrix M_{k+1} is obtained according to the transformation relations in the Bernstein basis:

$$a_{i,j} = \int_0^1 B_i\,(x) B_j(x) dx.$$

Computation of Integration Matrix M_{k+1}:

$$M_{k+1} = \begin{bmatrix} a_{00} & a_{01} \dots & a_{0n} \\ \vdots & \ddots \dots & \vdots \\ a_{n0} & a_{n1} \dots & a_{nn} \end{bmatrix}$$

$$M_{k+1} = [a_{i,j}]\,, \qquad i,j = 0,1,\dots,k.$$

Here, $a_{i,j}$ are the coefficients expressing the integral $\int_0^t B_{i,k}(x)dx$ in terms of the polynomials $B_{j,k}(t)$. These coefficients are calculated directly from the integrals of Bernstein polynomials and placed into the corresponding matrix elements.

Construction of the d vector:

$$d = [d_0, d_1, d_2, \dots, d_k]^T$$
$$d_i = \int_0^t B_{i,k}\,(t)dt \quad i = \{0,1,\dots,k\},\ t = [0,1]\,.$$

Construction of the 7×7 matrix for $k = 6$:

Bernstein Polynomials:

$$b_{i,6}(t) = \binom{6}{i} t^i (1-t)^{6-i} \quad , \quad i = 0,1,\dots,6.$$

Example polynomials:

$$b_{0,6}(t) = (1-t)^6$$
$$b_{1,6}(t) = 6t(1-t)^5$$
$$b_{2,6}(t) = 15t^2(1-t)^4$$
...
$$b_{6,6}(t) = t^6$$

Orthonormal Bernstein Polynomials

By applying the Gram-Schmidt method:
$B(t) = \left[B_{0,k}(t), B_{1,k}(t), \dots, B_{k,k}(t)\ \right]^T$.

Integration Matrix for Bernstein Polynomials

$$\int_0^t B(x)dx = M_7\, B(t).$$

Calculation of Integration Coefficients for Each Element

$$a_{i,j} = \int_0^1 B_i\,(x) B_j(x) dx.$$

Resulting Integration Matrix M_7:

$M_7 =$ [0.132653, 0.253433, 0.219333, 0.195022, 0.164421, 0.127498, 0.073561;
−0.009386, 0.112245, 0.219980, 0.175011, 0.152727, 0.116644, 0.067942;
0.001415, −0.016921, 0.091836, 0.184074, 0.129435, 0.109257, 0.059833;
−0.000340, 0.004070, −0.022088, 0.071429, 0.144884, 0.083131, 0.0584945;
0.000115, −0.001376 0.007467 −0.024147 0.051020 0.100996 0.036127;
−0.000049, 0.000592, −0.003213, 0.010391, −0.021956, 0.030612, 0.048603;
0.000021, −0.000256, 0.001391, −0.004499, 0.009507, −0.013255, 0.010204].

$d = [0.5151\ 0.4738\ 0.4286\ 0.3780\ 0.3194\ 0.2474\ 0.1429]^T$.

This matrix and vector are used for the integration of orthonormal Bernstein polynomials in the case $k = 6$.

Example 3. $f^{iv}(t) + 2a^2 f^{ii}(t) + a^4 f(t) = \cos at$

$f(0) = 0,\ f'(0) = 0,\ f''(0) = 0,\ f'''(0) = 0$ using the method Aboodh-Bernstein.

1. step Solution: Taking the Aboodh transform, we get

$A[f^{iv}(t)] + A[2a^2 f^{ii}(t)] + A[a^4 f(t)] = A[\cos at]$
$v^{iv}A(v) + 2a^2 v^2 A(v) + a^4 A(v) = \frac{1}{v^2+a^2}$
$A(v)[v^4 + 2a^2 v^2 + a^4] = \frac{1}{v^2+a^2}$
$A(v)(v^2 + a^2)^2 = \dfrac{1}{v^2 + a^2}$
$F(v) = \frac{1}{(v^2+a^2)^3}$
$f(t) = \dfrac{1}{2} t^2 e^{-at}$.

2. step Bernstein polynomial

$$F(v) = \frac{1}{(v^2+a^2)^3} \tag{1}$$

using this expression in Equation (1), we will create the matrix form of the integral operator and calculate the bit coefficient matrix and a series of specific values depending on this matrix.

$B_n(f,t) = f\left(\dfrac{k}{n}\right)\binom{n}{k} t^k (1-t)^{n-k}$

$C^T = d^T\left(\left(M_{k+1}^2 + a^2\right)^3\right)^{-1}$.

Here $a -$ real solution, $d -$ the matrix has a certain size.

Approximation solution

$f_k(t) = C^T B(t)$.

Table 2 presents the comparison between the exact and approximate solutions, while Table 3 provides the relative error analysis for Example 3. Figure 1 illustrates the behavior of the exact and approximate solutions in the interval $t \in [0,1]$.

Table 2. *Real Solution and Approximation*

t	Real Solution	Approximation	Error
0.0	0	0.0	0.0
0.1	0.00045	0.00044	8.39×10^{-9}
0.2	0.00161	0.00158	2.69×10^{-9}
0.3	0.00318	0.00311	2.87×10^{-10}
0.4	0.00529	0.00516	1.26×10^{-9}
0.5	0.00803	0.00782	2.06×10^{-9}
0.6	0.01151	0.01119	3.18×10^{-9}
0.7	0.01589	0.01542	4.7×10^{-9}
0.8	0.02133	0.02066	6.69×10^{-9}
0.9	0.02806	0.02714	9.25×10^{-10}
1.0	0.03627	0.03496	1.314×10^{-8}

Table 3. *Comparison of Relative Errors for Example 3*

t	Laplace-Bernstein	Sumudu-Bernstein [12]	Aboodh-Bernstein (proposed model)
0.0	0.000000	1.590240×10^{-7}	0.0
0.1	0.000351	4.158605×10^{-8}	8.39×10^{-10}
0.2	0.000536	5.120531×10^{-8}	2.69×10^{-9}
0.3	0.000273	2.741731×10^{-9}	2.87×10^{-9}
0.4	0.001450	4.740686×10^{-8}	1.26×10^{-9}
0.5	0.009641	5.364790×10^{-9}	2.06×10^{-9}
0.6	0.036447	4.664612×10^{-8}	3.18×10^{-9}
0.7	0.102983	7.488886×10^{-9}	4.7×10^{-9}
0.8	0.244894	5.191783×10^{-8}	6.69×10^{-9}
0.9	0.519323	3.474813×10^{-8}	9.25×10^{-10}
1.0	1.014333	1.590240×10^{-7}	1.314×10^{-8}

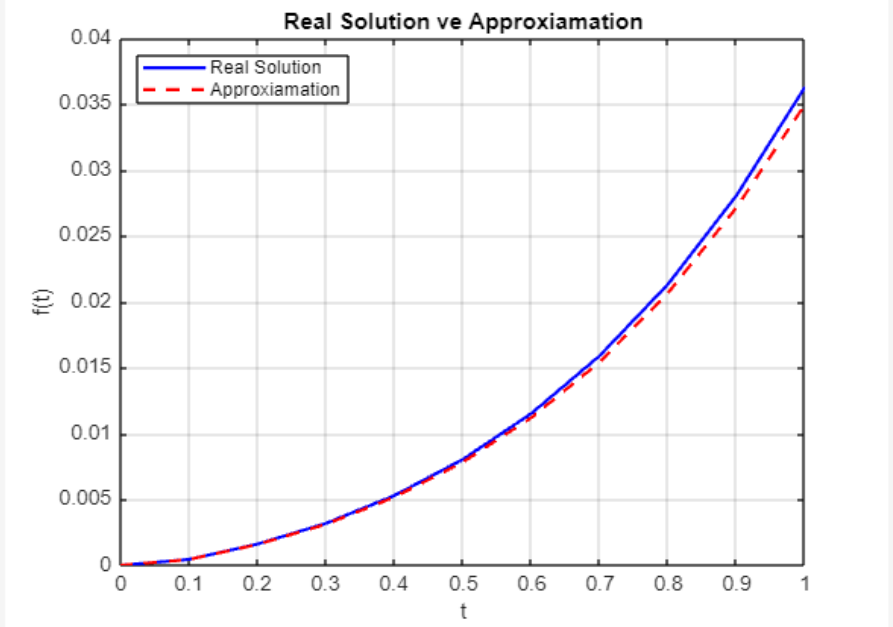


Figure 1. *Real solution and approximation for Example 3*

Example 4. Determine the solution for the linear Volterra integral equation

$\int_0^t cos(t - s)f(s)ds = t \sin t.$

Solution: Taking the Aboodh transform

$$A[\int_0^t cos(t\ -\ s)f(s)ds]\ = A[t\sin t]$$
$$F(v) = \frac{2v}{v^2+1}$$
$$f(t) = 2\sin t$$
$$F(M_{k+1}) = \frac{2M_{k+1}}{M_{k+1}^2+1}$$
$$C^T = d^T 2M_{k+1}(M_{k+1}^2+1)^{-1}$$
$$f_k(t) = C^T B(t).$$

$$t = 0.1$$
$$f_{\sim}(0.1) = B_0(0.1)f(0) + B_1(0.1)f(0.1) + \cdots\ + B_6(0.1)f(1)\,. \quad (2)$$

Using the formula in Equation (2), we can compute the approximate solution for each value of t in $[0,1]$. Table 4 presents the comparison between the exact and approximate solutions, while Table 5 provides the relative error analysis for Example 4. The behavior of both exact and approximate solutions within the interval $t \in [0,1]$ is shown in Figure 2.

Table 4. *Real Solution and Approximation*

t	Real Solution	Approximate Solution	Error
0.0	0.000000	0.000000001832443488	0.000000001832443488
0.1	0.198669	0.198669524	0.000000524
0.2	0.389418	0.389416236	0.000001764
0.3	0.564642	0.564636167	0.000005833
0.4	0.717356	0.717344165	0.000011835
0.5	0.841471	0.841450848	0.000020152
0.6	0.932039	0.932001012	0.000037988
0.7	0.985450	0.985380064	0.000069936
0.8	0.998058	0.997928094	0.000129906
0.9	0.968912	0.968686428	0.000225572
1	0.909297	0.90887776	0.00041924

Table 5. *Comparison of Relative Errors for Example 4*

t	Laplace-Bernstein	Sumudu-Bernstein [12]	Aboodh-Bernstein (proposed model)
0.0	0.0637	0.0000000012	$1.832443488 \times 10^{-9}$
0.1	0.1125	1.28×10^{-6}	5.24×10^{-7}
0.2	0.1283	2.84×10^{-6}	1.764×10^{-6}
0.3	0.1119	3.62×10^{-6}	5.833×10^{-6}
0.4	0.0644	4.14×10^{-6}	1.11835×10^{-5}
0.5	0.0129	4.75×10^{-6}	2.0152×10^{-5}
0.6	0.1187	5.11×10^{-6}	3.7988×10^{-5}
0.7	0.2514	6.48×10^{-6}	6.9936×10^{-5}
0.8	0.4099	7.24×10^{-6}	1.29906×10^{-4}
0.9	0.5929	8.11×10^{-6}	2.225572×10^{-4}
1	0.7992	8.98×10^{-6}	4.1924×10^{-4}

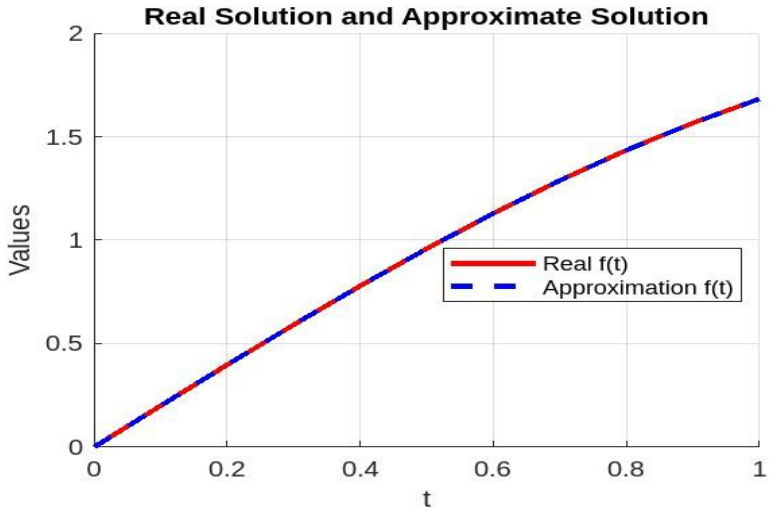


Figure 2. *Real solution and Approximate Solution for Example 4*

Example 5. Determine the solution for the nonlinear integral equation

$te^t = \int_0^t e^{(t-s)}\, e^{f(s)} ds$

with the exact solution $f(t) = t$.

Solution: Taking the Aboodh transform $a = 1$

$A[\mathrm{te^t}] = vA\left[\mathrm{e^{(t-s)}}\right]A\left[\mathrm{e^{f(t)}}\right]$

$\frac{1}{v(v-a)^2} = \frac{v}{(v^2-v)}A\left[\mathrm{e^{f(t)}}\right]$

$\frac{1}{v(v-1)} = A\left[\mathrm{e^{f(t)}}\right]$

$e^{f(t)} = A^{-1}F(v)$

$F(v) = \frac{1}{v(v-1)}.$

$e^{f(t)} = X(t)$ the solution will be $f_k(t) = lnX(t).$

The operational matrix of integration is

$F[M_{k+1}] = \frac{1}{{M_{k+1}}^2 - M_{k+1}}$

$M_{k+1} = (k+1) \times (k+1)$ operator matrix.

The coefficient matrix is

$C^T = d^T({M_{k+1}}^2 - M_{k+1})^{-1}$

$X(t) = \sum_{i=0}^{n} C_i\, B_{i,n}(t) = C^T B(t).$

For $k = 6$, we obtain the following

$f_k(t) = lnX(t).$

The exact and approximate solutions are reported in Table 6, while Table 7 provides a comparative analysis of the corresponding relative errors. Figure 3 illustrates the true and approximate solutions for Example 5.

Table 6. *Real Solution and Approximation*

t	$X(t)$	$f_k(t) = \ln(X(t))$	$f(t) = t$	$Error(t) = \| f_k(t) - f(t) \|$
0.0	1.000000	0.000000	0.000000	0.000000
0.2	1.221542	0.199988	0.200000	0.000012
0.4	1.491213	0.399796	0.400000	0.000204
0.6	1.822264	0.599253	0.600000	0.000747
0.8	2.225350	0.799323	0.800000	0.000677
1	2.718576	1.000104	1.000000	0.000104

Table 7. *Comparison of Relative Errors for Example 5*

t	Laplace-Bernstein	Aboodh-Bernstein (proposed model)
0.0	0.0644	0.000000
0.2	0.1361	0.000012
0.4	0.0778	0.000204
0.6	0.1099	0.000747
0.8	0.4286	0.000677
1	0.8571	0.000104

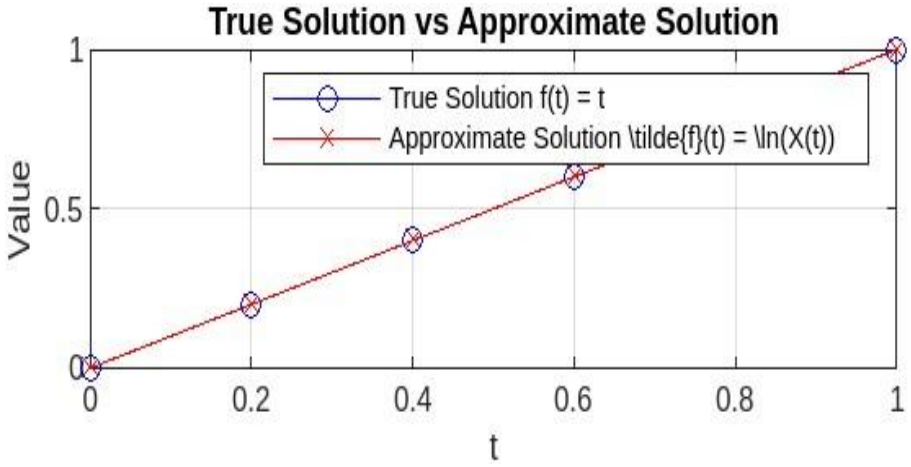


Figure 3. *True Solution vs Approximate Solution for Example 5*

Example 6. Discover the solution to the nonlinear integral equation

$$\frac{1}{2}t^2 - \frac{1}{3}t^3 + \frac{1}{12}t^4 = \int_0^t (t-s)f^2(s)ds.$$

Exact solution $f(t) = 1 + t$.

Solution: Using Aboodh Transform

$$\frac{1}{2}A[t^2] - \frac{1}{3}A[t^3] + \frac{1}{12}A[t^4] = A[\int_0^t (t-s)f^2(s)ds]$$
$$\frac{1}{2}A[t^2] - \frac{1}{3}A[t^3] + \frac{1}{12}A[t^4] = vA[t-s]A[f^2(s)]$$
$$\frac{1}{v^4} - \frac{2}{v^5} + \frac{2}{v^6} = \frac{1}{v^2}A[f^2(t)]$$
$$\frac{1}{v^2} - \frac{2}{v^3} + \frac{2}{v^4} = A[f^2(t)]$$

$$F(v) = \frac{1}{v^2} - \frac{2}{v^3} + \frac{2}{v^4}.$$

$$f^2(t) = A^{-1}[F(v)]\ , \qquad f^2(t) = X(t)\ , \qquad f_k(t) = \sqrt{X(t).}$$

The operational matrix of integration is

$$F(M_{k+1}) = M_{k+1}^{-2} - 2M_{k+1}^{-3} + 2M_{k+1}^{-4}$$
$$C^T = d^T(M_{k+1}^{-2} - 2M_{k+1}^{-3} + 2M_{k+1}^{-4})$$

$$B_k(t) = \binom{n}{k}(1-t)^{n-k}t^k$$
$$X(t) = C^T B(t)\ , \qquad f_k(t) = \sqrt{X(t).}$$

The exact and approximate solutions are listed in Table 8, with Table 9 showing the comparative analysis of their relative errors. The true solution versus the approximate solution for Example 6 is illustrated in Figure 4.

Table 8. *Real Solution and Approximation*

t	$f_k(t) = \sqrt{X(t)}$	$f(t)$	Errors
0.0	1.0000000000000000	1.0	0
0.1	1.1000000000000001	1.1	0
0.2	1.2000000000000000	1.2	0
0.3	1.2999999999999996	1.3	$4 * 10^{-15}$
0.4	1.3999999999999981	1.4	$1.8 * 10^{-15}$
0.5	1.4999999999999953	1.5	$4.7* 10^{-15}$
0.6	1.5999999999999901	1.6	$1 * 10^{-15}$
0.7	1.6999999999999826	1.7	$1.73 * 10^{-13}$
0.8	1.7999999999999723	1.8	$2.78 * 10^{-13}$
0.9	1.8999999999999595	1.9	$4.04 * 10^{-13}$
1	1.9999999999999458	2.0	$5.42 * 10^{-13}$

Table 9. *Comparison of Relative Errors for Example 6*

t	Laplace-Bernstein [5]	Sumudu-Bernstein [12]	Aboodh-Bernstein (proposed model)
0.0	0.000	0	0
0.1	0.005	0	0
0.2	0.011	$1.110223 * 10^{-15}$	0
0.3	0.019	$3.108625 * 10^{-15}$	$4 * 10^{-15}$
0.4	0.025	$7.993606 * 10^{-15}$	$1.8 * 10^{-15}$
0.5	0.044	$1.110223 * 10^{-15}$	$4.7* 10^{-15}$
0.6	0.050	$9.992007 * 10^{-15}$	$1 * 10^{-15}$
0.7	0.036	$5.107026 * 10^{-15}$	$1.73 * 10^{-13}$
0.8	0.002	$2.287059 * 10^{-15}$	$2.78 * 10^{-13}$
0.9	0.001	$3.819167 * 10^{-15}$	$4.04 * 10^{-13}$
1	0.00001	$5.306866 * 10^{-15}$	$5.42 * 10^{-13}$

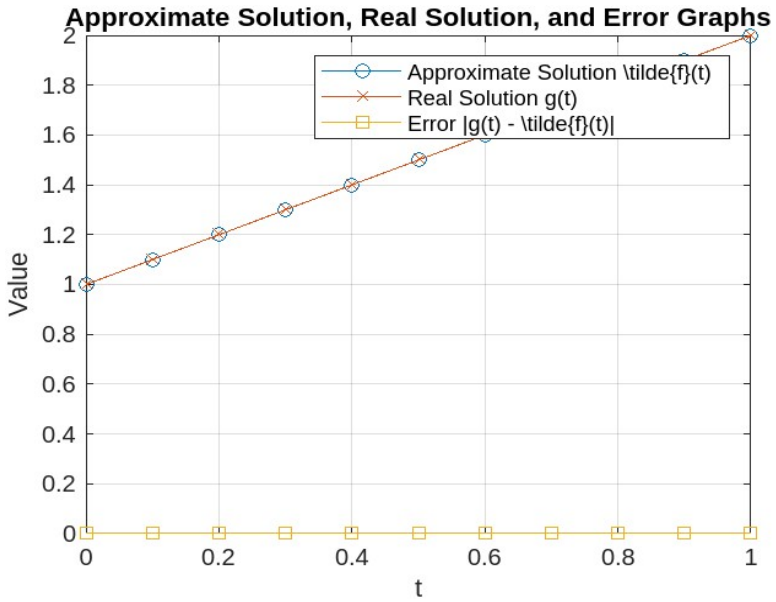


Figure 4. *Approximate Solution, Real Solution and Error Graphs for Example 6*

4. RESULTS

The analysis of the table and graphs facilitated a comprehensive evaluation of the effectiveness of approximating the real function using Bernstein polynomials. The table showcases the real function, its approximation, and the corresponding error for each value within the interval [0,1]. Generally, low error values indicate a close approximation to the real function, while regions with higher error values highlight areas of potential inaccuracy.

The graphical representation further substantiates this assessment. In the first subplot, the real function and its approximation are juxtaposed. A close alignment of the curves signifies a successful emulation of the real function by the approximation.

In conclusion, the analysis demonstrates the overall effectiveness of Bernstein polynomial approximation, while also identifying areas for potential improvement. By optimizing the degree of the polynomials and the number of nodes, more accurate approximations can be achieved, though this may increase computational complexity. This underscores the utility of Bernstein polynomials in numerical analysis, particularly for approximating complex functions where analytical solutions are difficult to obtain.

Furthermore, the integration of the Aboodh transformation enhances this approach. The Aboodh transformation refines the Bernstein polynomial method by identifying optimal node points, thereby providing more precise and efficient approximations. This transformation is particularly advantageous for reducing error rates without increasing the polynomial degree. Consequently, the combination of Bernstein polynomials and the Aboodh transformation offers a robust and effective method, balancing high accuracy with manageable computational demands. This makes the approach widely applicable, enabling more effective approximations of various functions in diverse contexts.

The Aboodh-Bernstein method has been developed and applied to solve a variety of equations, including differential and integral equations. Through numerical examples, the method has been demonstrated on linear differential equations and nonlinear integral equations, with a specific approach for addressing the nonlinear components. The absolute errors between the exact and approximate solutions for Examples 1 to 4, as presented in Tables 2 to 8, and the corresponding graphs for Examples 1 to 4, shown in Figures 1 to 4, illustrate that the method achieves a high precision with only a few polynomial terms. Furthermore, a comparison with an existing method, as detailed in Table 3 to 9 , shows that the Aboodh-Bernstein method

provides superior accuracy. It delivers better results compared to the Laplace transform [5] and performs on par with the Sumudu transform [12]. All computations were executed using MATLAB 2024.

The results indicate that the Aboodh-Bernstein method not only improves the accuracy of numerical solutions for differential and integral equations but also demonstrates scalability for more complex systems. Future studies can explore the application of this method to fractional-order equations, systems with boundary layer behavior, or problems with discontinuous initial conditions. Moreover, incorporating other transforms such as Elzaki or Mohand transform may further expand its capability [27].

It can be suggested to future researchers that the method applied in the paper can be tested with (p,q)-Berstein operators [28,29], Bernstein Schurer [30], (p,q)-Bernstein Schurer [31], Bernstein Stancu [32] and (p,q)-Bernstein Stancu [33] polynomials instead of Bernstein polynomials and that the study can be investigated for different types of equations based on [34].

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CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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APPENDIX

The following MATLAB code illustrates the implementation of the Aboodh-Bernstein method for solving differential or integral equations. It includes the steps for computing the Bernstein basis functions, the operational integration matrix M_{k+1}, applying the Aboodh transform, and generating the approximate solution:

Parameters

```
syms v t a
k = 6;
a = 1;
n = k + 1;
t_vals = linspace(0,1,11);
```

1. Define Bernstein polynomials

```
B = sym(zeros(1,n));
for i = 0:k
   B(i+1) = nchoosek(k,i)*t^i*(1 - t)^(k - i);
end
```

2. Compute M_{k+1} integration matrix

```
M = sym(zeros(n,n));
for i = 1:n
   for j = 1:n
      integrand = B(j) * int(B(i), t, 0, t);
```

```
      M(i,j) = int(integrand, t, 0, 1);
   end
end
M = double(M);
```

3. Apply Aboodh Transform (example: A[cos(at)] case)

```
F_v = 1 / (v^2 + a^2)^3;
F_M = subs(F_v, v, M);
F_M = double(F_M);
```

4. Compute d vector

```
d = zeros(1, n);
for i = 1:n
   d(i) = double(int(B(i), t, 0, 1));
end
d = d';
```

5. Compute coefficient vector C^T

```
C_T = d.' * F_M;
```

6. Approximate solution f_k(t)

```
f_k = zeros(size(t_vals));
for idx = 1:length(t_vals)
   t_num = t_vals(idx);
   Bt = zeros(n,1);
   for i = 0:k
      Bt(i+1) = double(subs(B(i+1), t, t_num));
   end
   f_k(idx) = C_T * Bt;
end
```

7. Compare with exact solution and compute error

```
f_real = 0.5 * t_vals.^2 .* exp(-a * t_vals);
err = abs(f_real - f_k);
```

8. Display and Plot

```
disp('t      Real      Approx     Error')
disp([t_vals' f_real' f_k' err'])

figure;
plot(t_vals, f_real, 'b-o', 'DisplayName', 'Exact');
hold on;
plot(t_vals, f_k, 'r--s', 'DisplayName', 'Approximation');
legend; grid on;
title('Exact vs Approximate Solution');
xlabel('t'); ylabel('f(t)');
```