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Research Article

Determinant and Inverse of Meet and Join Matrices

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We define meet and join matrices on two subsets X and Y of a lattice $(P,\preceq)$ with respect to a complex-valued function f on P by $(X,Y)_f = (f(x_i \wedge y_i))$ and $[X,Y]_f = (f(x_i \vee y_i))$, respectively. We present expressions for the determinant and the inverse of $(X,Y)_f$ and $[X,Y]_f$, and as special cases we obtain several new and known formulas for the determinant and the inverse of the usual meet and join matrices $(S)_f$ and $[S]_f$.

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1. Introduction

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers, and let f be an arithmetical function. Let $(S)_f$ denote the $n \times n$ matrix having f evaluated at the greatest common divisor (x_i, x_j) of x_i and x_j as its ij-entry, that is, $(S)_f = (f((x_i, x_j)))$. Analogously, let $[S]_f$ denote the $n \times n$ matrix having f evaluated at the least common multiple $[x_i, x_j]$ of x_i and x_j as its ij-entry, that is, $[S]_f = (f([x_i, x_j]))$. The matrices $(S)_f$ an $[S]_f$ are referred to as the GCD and LCM matrices on S associated with f, respectively. The set S is said to be factor-closed if it contains every divisor of x for any $x \in S$, and the set S is said to be GCD-closed if $(x_i, x_j) \in S$ whenever $x_i, x_j \in S$. Every factor-closed set is GCD-closed but the converse does not hold.

Smith [1] calculated $\det(S)_f$ when S is factor-closed and $\det[S]_f$ in a more special case. Since Smith, a large number of results on GCD and LCM matrices have been presented in the literature. For general accounts, see, for example, [2–5].

Let $(P,\preceq)$ be a lattice in which every principal order ideal is finite. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a subset of P, and let f be a complex-valued function on P. Then the $n \times n$ matrix $(S)_f = (f(x_i \wedge x_j))$ is called the meet matrix on S associated with f and the $n \times n$ matrix

$[S]_f = (f(x_i \vee x_j))$ is called the join matrix on S associated with f. If $(P, \preceq) = (\mathbb{Z}^+, |)$, then meet and join matrices, respectively, become GCD and LCM matrices. The set S is said to be lower-closed (resp., upper-closed) if for every $x, y \in P$ with $x \in S$ and $y \preceq x$ (resp., $x \preceq y$), we have $y \in S$. The set S is said to be meet-closed (resp., join-closed) if for every $x, y \in S$, we have $x \wedge y \in S$ (resp., $x \vee y \in S$).

Meet matrices have been studied in many papers (see, e.g., [3, 5–11]). Haukkanen [7] calculated the determinant of $(S)_f$ on arbitrary set S and obtained the inverse of $(S)_f$ on a lower-closed set S. Korkee and Haukkanen [12] obtained the inverse of $(S)_f$ on a meet-closed set S.

Join matrices have previously been studied by Hong and Sun [13], Korkee and Haukkanen [5], and Wang [11]. Korkee and Haukkanen [5] present, among others, formulas for the determinant and inverse of $[S]_f$ on meet-closed, join-closed, lower-closed, and upper-closed sets S.

Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be two subsets of P. We define the meet matrix on X and Y with respect to f as $(X, Y)_f = (f(x_i \wedge y_j))$. In particular, $(S, S)_f = (S)_f$. Analogously, we define the join matrix on X and Y with respect to f as $[X, Y]_f = (f(x_i \vee y_j))$. In particular, $[S, S]_f = [S]_f$.

In this paper we present expressions for the determinant and the inverse of $(X, Y)_f$ on arbitrary sets X and Y. If $X = Y = S$, then we obtain the determinant formula for $(S)_f$ given in [7] and a formula for the inverse of $(S)_f$ on arbitrary set S. If S is meet-closed or lower-closed, then the formula for the inverse of $(S)_f$ reduces to those given in [7, 12]. We also obtain a new expression for the inverse formulas of $(S)_f$ given in [7, 12].

We also present expressions for the determinant and inverse of $[X, Y]_f$ when the function f is semimultiplicative (for definition, see (6.1)). As special cases, we obtain formulas for the determinant and inverse of $[S]_f$ on arbitrary set S. These formulas generalize the determinant and the inverse formulas of $[S]_f$ on meet-closed and lower-closed sets S presented in [5]. As special cases, we also obtain some new and known results on LCM matrices.

Determinant and inverse formulas for $(S)_f$ and $[S]_f$ on join-closed and upper-closed sets S could be obtained applying duality to the results of this paper. We do not include the details of these results here.

2. Preliminaries

Let $(P, \preceq)$ be a lattice in which every principal order ideal is finite, and let f be a complex-valued function on P. Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be two subsets of P. Let the elements of X and Y be arranged so that $x_i \preceq x_j \Rightarrow i \leq j$ and $y_i \preceq y_j \Rightarrow i \leq j$. Let $D = \{d_1, d_2, \ldots, d_m\}$ be any subset of P containing the elements $x_i \wedge y_j$, $i, j = 1, 2, \ldots, n$. Let the elements of D be arranged so that $d_i \preceq d_j \Rightarrow i \leq j$. We define the function $\Psi_{D,f}$ on D inductively as

$$\Psi_{D,f}(d_k) = f(d_k) - \sum_{d_v \prec d_k} \Psi_{D,f}(d_v) \tag{2.1}$$

or

$$f(d_k) = \sum_{d_v \preceq d_k} \Psi_{D,f}(d_v). \tag{2.2}$$

Then

$$\Psi_{D,f}(d_k) = \sum_{d_v \preceq d_k} f(d_v)\mu_D(d_v, d_k), \tag{2.3}$$

where μ_D is the Möbius function of the poset $(D, \preceq)$, see [14, Section IV.1]. If D is meet-closed, then

$$\Psi_{D,f}(d_k) = \sum_{\substack{z \preceq d_k \\ z \not\preceq d_t \\ t<k}} \sum_{w \preceq z} f(w)\mu(w, z), \tag{2.4}$$

where μ is the Möbius function of P, and if D is lower-closed, then

$$\Psi_{D,f}(d_k) = \sum_{d_v \preceq d_k} f(d_v)\mu(d_v, d_k), \tag{2.5}$$

where μ is the Möbius function of P. For proofs of (2.4) and (2.5), see [7]. If $(P, \preceq) = (\mathbb{Z}^+, |)$ and D is factor-closed, then $\mu_D(d_v, d_k) = \mu(d_k/d_v)$ (see [15, Chapter 7]), where μ is the number-theoretic Möbius function, and (2.3) becomes

$$\Psi_{D,f}(d_k) = \sum_{d_v | d_k} f(d_v)\mu\left(\frac{d_k}{d_v}\right) = (f * \mu)(d_k), \tag{2.6}$$

where $*$ is the Dirichlet convolution of arithmetical functions.

Let $E(X) = (e_{ij}(X))$ and $E(Y) = (e_{ij}(Y))$ denote the $n \times m$ matrices defined by

$$\begin{aligned} e_{ij}(X) &= \begin{cases} 1 & \text{if } d_j \preceq x_i, \\ 0 & \text{otherwise,} \end{cases} \\ e_{ij}(Y) &= \begin{cases} 1 & \text{if } d_j \preceq y_i, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{2.7}$$

respectively. Note that $E(X)$ and $E(Y)$ depend on D but for the sake of brevity, D is omitted from the notation. We also denote

$$\Lambda_{D,f} = \operatorname{diag}(\Psi_{D,f}(d_1), \Psi_{D,f}(d_2), \ldots, \Psi_{D,f}(d_m)). \tag{2.8}$$

3. A structure theorem

In this section, we give a factorization of the matrix $(X, Y)_f = (f(x_i \wedge y_j))$. As special cases, we obtain the factorizations of $(S)_f$ given in [7, 9, 12]. A large number of similar factorizations are presented in the literature. The idea of this kind of factorization may be considered to originate from Pólya and Szegö [16].

Theorem 3.1. *One has*

$$(X,Y)_f = E(X)\Lambda_{D,f}E(Y)^T. \tag{3.1}$$

Proof. By (2.2), the ij-entry of $(X,Y)_f$ is

$$f(x_i \wedge y_j) = \sum_{d_v \preceq x_i \wedge y_j} \Psi_{D,f}(d_v). \tag{3.2}$$

Now, applying (2.7) and (2.8) to (3.2), we obtain Theorem 3.1. □

Remark 3.2. The sets X and Y could be allowed to have distinct cardinalities in Theorems 3.1 and 6.1. However, in other results we must assume that these cardinalities coincide.

4. Determinant formulas

In this section, we derive formulas for determinants of meet matrices. In Theorem 4.1, we present an expression for $\det(X,Y)_f$ on arbitrary sets X and Y. Taking $X = Y = S = \{x_1, x_2, \ldots, x_n\}$ in Theorem 4.1, we could obtain a formula for the determinant of usual meet matrices $(S)_f$ on arbitrary set S (see [7, Theorem 3]), and further taking $(P, \preceq) = (\mathbb{Z}^+, |)$ we could obtain a formula for the determinant of GCD matrices on arbitrary set S (see [17, Theorem 2]). In Theorems 4.2 and 4.4, respectively, we calculate $\det(S)_f$ on meet-closed and lower-closed sets S. These formulas are also given in [7].

Theorem 4.1. (i) *If $n > m$, then* $\det(X,Y)_f = 0$.

(ii) *If $n \le m$, then*

$$\det(X,Y)_f = \sum_{1 \le k_1 < k_2 < \cdots < k_n \le m} \det E(X)_{(k_1,k_2,\ldots,k_n)} \det E(Y)_{(k_1,k_2,\ldots,k_n)} \times \Psi_{D,f}(d_{k_1})\Psi_{D,f}(d_{k_2}) \cdots \Psi_{D,f}(d_{k_n}). \tag{4.1}$$

Proof. By Theorem 3.1,

$$\det(X,Y)_f = \det\left(E(X)\Lambda_{D,f}E(Y)^T\right). \tag{4.2}$$

Thus by the Cauchy-Binet formula, we obtain Theorem 4.1. □

Theorem 4.2. *If S is meet-closed, then*

$$\det(S)_f = \prod_{v=1}^{n} \Psi_{S,f}(x_v) = \prod_{v=1}^{n} \sum_{\substack{z \preceq x_v \\ z \not\preceq x_t \\ t<v}} \sum_{w \preceq z} f(w)\mu(w,z). \tag{4.3}$$

Proof. We take $X = Y = S$ in Theorem 4.1. Since S is meet-closed, we may further take $D = S$. Then $m = n$ and $\det E(S)_{(k_1,k_2,\ldots,k_n)} = \det E(S)_{(1,2,\ldots,n)} = 1$, and so we obtain the first equality in (4.3). The second equality follows from (2.4). □

Remark 4.3. Theorem 4.2 can also be proved by taking $X = Y = S$ and $D = S$ in Theorem 3.1.

Theorem 4.4. *If S is lower-closed, then*

$$\det(S)_f = \prod_{v=1}^{n} \Psi_{S,f}(x_v) = \prod_{v=1}^{n} \sum_{x_u \preceq x_v} f(x_u)\mu(x_u, x_v). \tag{4.4}$$

Proof. The first equality in (4.4) follows from (4.3). The second equality follows from (2.5). □

Corollary 4.5 [18, Theorem 2]. *Let S be a GCD-closed set of distinct positive integers, and let f be an arithmetical function. Then*

$$\det(S)_f = \prod_{v=1}^{n} \sum_{\substack{z \mid x_v \\ z \nmid x_t \\ t<v}} (f * \mu)(z). \tag{4.5}$$

Corollary 4.6 [1]. *Let S be a factor-closed set of distinct positive integers, and let f be an arithmetical function. Then*

$$\det(S)_f = \prod_{v=1}^{n} (f * \mu)(x_v). \tag{4.6}$$

5. Inverse formulas

In this section, we derive formulas for inverses of meet matrices. In Theorem 5.1, we present an expression for the inverse of $(X, Y)_f$ on arbitrary sets X and Y, and in Theorem 5.2 we present an expression for the inverse of $(S)_f$ on arbitrary set S. Taking $(P, \preceq) = (\mathbb{Z}^+, |)$, we could obtain a formula for the inverse of GCD matrices on arbitrary set S. Formulas for the inverse of meet or GCD matrices on arbitrary set have not previously been presented in the literature. In Theorems 5.3 and 5.5, respectively, we calculate the inverse of $(S)_f$ on meet-closed and lower-closed sets S. Similar formulas are given in [12, Theorem 7.1] and [7, Theorem 6].

Theorem 5.1. *Let $X_i = X \setminus \{x_i\}$ and $Y_i = Y \setminus \{y_i\}$ for $i = 1, 2, \ldots, n$. If $(X, Y)_f$ is invertible, then the inverse of $(X, Y)_f$ is the $n \times n$ matrix $B = (b_{ij})$, where*

$$b_{ij} = \frac{(-1)^{i+j}}{\det(X, Y)_f} \sum_{1 \le k_1 < k_2 < \cdots < k_{n-1} \le m} \det E(X_j)_{(k_1, k_2, \ldots, k_{n-1})} \det E(Y_i)_{(k_1, k_2, \ldots, k_{n-1})} \times \Psi_{D,f}(d_{k_1}) \Psi_{D,f}(d_{k_2}) \cdots \Psi_{D,f}(d_{k_{n-1}}). \tag{5.1}$$

Proof. It is well known that

$$b_{ij} = \frac{\alpha_{ji}}{\det(X, Y)_f}, \tag{5.2}$$

where α_{ji} is the cofactor of the ji-entry of $(X,Y)_f$. It is easy to see that $\alpha_{ji} = (-1)^{i+j}\det(X_j, Y_i)_f$. By Theorem 4.1, we see that

$$\det(X_j,Y_i)_f = \sum_{1\le k_1<k_2<\cdots<k_{n-1}\le m} \det E(X_j)_{(k_1,k_2,\ldots,k_{n-1})} \det E(Y_i)_{(k_1,k_2,\ldots,k_{n-1})} \times \Psi_{D,f}(d_{k_1})\Psi_{D,f}(d_{k_2})\cdots\Psi_{D,f}(d_{k_{n-1}}). \tag{5.3}$$

Combining the above equations, we obtain Theorem 5.1. □

Theorem 5.2. *Let $S_i = S\setminus\{x_i\}$ for $i = 1,2,\ldots,n$. If $(S)_f$ is invertible, then the inverse of $(S)_f$ is the $n\times n$ matrix $B = (b_{ij})$, where*

$$b_{ij} = \frac{(-1)^{i+j}}{\det(S)_f} \sum_{1\le k_1<k_2<\cdots<k_{n-1}\le m} \det E(S_j)_{(k_1,k_2,\ldots,k_{n-1})} \det E(S_i)_{(k_1,k_2,\ldots,k_{n-1})} \times \Psi_{D,f}(d_{k_1})\Psi_{D,f}(d_{k_2})\cdots\Psi_{D,f}(d_{k_{n-1}}). \tag{5.4}$$

Proof. Taking $X = Y = S$ in Theorem 5.1, we obtain Theorem 5.2. □

Theorem 5.3. *Suppose that S is meet-closed. If $(S)_f$ is invertible, then the inverse of $(S)_f$ is the $n\times n$ matrix $B = (b_{ij})$, where*

$$b_{ij} = \sum_{k=1}^{n} \frac{(-1)^{i+j}}{\Psi_{S,f}(x_k)} \det E(S_i^k)\det E(S_j^k), \tag{5.5}$$

where $E(S_i^k)$ is the $(n-1)\times(n-1)$ submatrix of $E(S)$ obtained by deleting the ith row and the kth column of $E(S)$, or

$$b_{ij} = \sum_{x_i\vee x_j\preceq x_k} \frac{\mu_S(x_i,x_k)\mu_S(x_j,x_k)}{\Psi_{S,f}(x_k)}, \tag{5.6}$$

where μ_S is the Möbius function of the poset $(S,\preceq)$.

Proof. Since S is meet-closed, we may take $D = S$. Then $E(S)$ is a square matrix with $\det E(S) = 1$. Further, $E(S)^T$ is the matrix associated with the zeta function of the finite poset $(S,\preceq)$. Thus the inverse of $E(S)^T$ is the matrix associated with the Möbius function of $(S,\preceq)$, that is, if $U = (u_{ij})$ is the inverse of $E(S)^T$, then $u_{ij} = \mu_S(x_i,x_j)$, see [14, Section IV.1]. On the other hand, $u_{ij} = \beta_{ij}/\det E(S)^T = \beta_{ij}$, where β_{ij} is the cofactor of the ij-entry of $E(S)$. Here $\beta_{ij} = (-1)^{i+j}\det E(S_i^j)$. Thus

$$(-1)^{i+j}\det E(S_i^j) = \mu_S(x_i,x_j). \tag{5.7}$$

Now we apply Theorem 5.2 with $D = S$. Then $m = n$, and using formulas(4.3) and (5.7), we obtain Theorem 5.3. □

Remark 5.4. Equation (5.6) is given in [7, Theorem 6] and can also be proved by taking $X = Y = S$ and $D = S$ in Theorem 3.1 and then applying the formula $(S)_f^{-1} = (E(S)^T)^{-1}\Lambda_{S,f}^{-1}E(S)^{-1}$.

Theorem 5.5. *Suppose that S is lower-closed. If $(S)_f$ is invertible, then the inverse of $(S)_f$ is the $n \times n$ matrix $B = (b_{ij})$, where*

$$b_{ij} = \sum_{x_i \vee x_j \preceq x_k} \frac{\mu(x_i, x_k)\mu(x_j, x_k)}{\Psi_{S,f}(x_k)}. \tag{5.8}$$

Here μ is the Möbius function of $(P, \preceq)$.

Proof. Since S is lower-closed, we have $\mu_S = \mu$ on S, (apply [14, Proposition 4.6]). Thus, Theorem 5.5 follows from Theorem 5.3. □

Corollary 5.6 [18, Corollary 1]. *Let S be a factor-closed set of distinct positive integers, and let f be an arithmetical function. If $(S)_f$ is invertible, then the inverse of $(S)_f$ is the $n \times n$ matrix $B = (b_{ij})$, where*

$$b_{ij} = \sum_{[x_i, x_j] | x_k} \frac{\mu(x_k/x_i)\mu(x_k/x_j)}{(f * \mu)(x_k)}. \tag{5.9}$$

Here μ is the number-theoretic Möbius function.

6. Formulas for join matrices

Let f be a complex-valued function on P. We say that f is a semimultiplicative function if

$$f(x)f(y) = f(x \wedge y)f(x \vee y) \tag{6.1}$$

for all $x, y \in P$ (see [5]).

The notion of a semimultiplicative function arises from the theory of arithmetical functions. Namely, an arithmetical function f is said to be semimultiplicative if $f(r)f(s) = f((r,s))f([r,s])$ for all $r, s \in \mathbb{Z}^+$. For semimultiplicative arithmetical functions, reference is made to the book by Sivaramakrishnan [19], see also [2]. Note that a semimultiplicative arithmetical function f with $f(1) \neq 0$ is referred to as a quasimultiplicative arithmetical function. Quasimultiplicative arithmetical functions with $f(1) = 1$ are the usual multiplicative arithmetical functions.

In this section, we show that join matrices $[X, Y]_f$ with respect to semimultiplicative functions f possess properties similar to those given for meet matrices $(X, Y)_f$ with respect to arbitrary functions f in Sections 3, 4, and 5. *Throughout this section, f is a semimultiplicative function on P such that $f(x) \neq 0$ for all $x \in P$.*

Theorem 6.1. *One has*

$$[X, Y]_f = \Delta_{X,f}(X, Y)_{1/f}\Delta_{Y,f} \tag{6.2}$$

or

$$[X, Y]_f = \Delta_{X,f}E(X)\Lambda_{D,1/f}E(Y)^T\Delta_{Y,f}, \tag{6.3}$$

where

$$\Delta_{X,f} = \operatorname{diag}\left(f(x_1), f(x_2), \ldots, f(x_n)\right). \tag{6.4}$$

Proof. By (6.1), the ij-entry of $[X,Y]_f$ is

$$f(x_i \vee y_j) = f(x_i)\frac{1}{f(x_i \wedge y_j)} f(y_j). \tag{6.5}$$

We thus obtain (6.2), and applying Theorem 3.1 we obtain (6.3). □

From (6.2), we obtain

$$\begin{aligned} \det[X,Y]_f &= \left(\prod_{v=1}^{n} f(x_v) f(y_v)\right) \det(X,Y)_{1/f}, \\ [X,Y]_f^{-1} &= \Delta_{Y,f}^{-1} (X,Y)_{1/f}^{-1} \Delta_{X,f}^{-1}. \end{aligned} \tag{6.6}$$

Now, using (6.6) and the formulas of Sections 4 and 5, we obtain formulas for join matrices.

We first present formulas for the determinant of join matrices. In Theorem 6.2, we give an expression for $\det[X,Y]_f$ on arbitrary sets X and Y. Formulas for the determinant of join or LCM matrices on arbitrary sets have not previously been presented in the literature. In Theorems 6.3 and 6.4, respectively, we calculate $\det[S]_f$ on meet-closed and lower-closed sets S. Similar formulas are given in [5, Section 5.3].

Theorem 6.2. (i) *If $n > m$, then* $\det[X,Y]_f = 0$.

(ii) *If $n \le m$, then*

$$\begin{aligned} \det[X,Y]_f = \left(\prod_{v=1}^{n} f(x_v) f(y_v)\right) \Bigg(&\sum_{1 \le k_1 < k_2 < \cdots < k_n \le m} \det E(X)_{(k_1,k_2,\ldots,k_n)} \det E(Y)_{(k_1,k_2,\ldots,k_n)} \\ &\times \Psi_{D,1/f}(d_{k_1}) \Psi_{D,1/f}(d_{k_2}) \cdots \Psi_{D,1/f}(d_{k_n}) \Bigg). \end{aligned} \tag{6.7}$$

Theorem 6.3. *if S is meet-closed, then*

$$\det[S]_f = \prod_{v=1}^{n} f(x_v)^2 \psi_{s,1/f}(x_v) = \prod_{v=1}^{n} f(x_v)^2 \sum_{\substack{z \preceq x_v \\ z \not\preceq x_t \\ t<v}} \sum_{w \preceq z} \frac{\mu(w,z)}{f(w)}. \tag{6.8}$$

Theorem 6.4. *If S is lower-closed, then*

$$\det[S]_f = \prod_{v=1}^{n} f(x_v)^2 \Psi_{S,1/f}(x_v) = \prod_{v=1}^{n} f(x_v)^2 \sum_{x_u \preceq x_v} \frac{\mu(x_u,x_v)}{f(x_u)}. \tag{6.9}$$

Corollary 6.5 [2, Theorem 3.2]. *Let S be a GCD-closed set of distinct positive integers, and let f be a quasimultiplicative arithmetical function such that $f(r) \neq 0$ for all $r \in \mathbb{Z}^+$. Then*

$$\det[S]_f = \prod_{v=1}^{n} f(x_v)^2 \sum_{\substack{z|x_v \\ z\nmid x_t \\ t<v}} \left(\frac{1}{f} * \mu\right)(z). \tag{6.10}$$

Corollary 6.6 [20, Theorem 2]. *Let S be a factor-closed set of distinct positive integers, and let f be a quasimultiplicative arithmetical function such that $f(r) \neq 0$ for all $r \in \mathbb{Z}^+$. Then*

$$\det[S]_f = \prod_{v=1}^{n} f(x_v)^2 \left(\frac{1}{f} * \mu\right)(x_v). \tag{6.11}$$

We next derive formulas for inverses of join matrices. In Theorem 6.7, we give an expression for the inverse of $[X,Y]_f$ on arbitrary sets X and Y, and in Theorem 6.8 we give an expression for the inverse of $[S]_f$ on arbitrary set S. Taking $(P,\preceq) = (\mathbb{Z}^+,|)$ we could obtain a formula for the inverse of LCM matrices on arbitrary set S. Formulas for the inverse of join or LCM matrices on arbitrary set have not previously been presented in the literature. In Theorems 6.9 and 6.10, respectively, we calculate the inverse of $[S]_f$ on meet-closed and lower-closed sets S. Similar formulas are given in [5, Section 5.3].

Theorem 6.7. *Let $X_i = X \setminus \{x_i\}$ and $Y_i = Y \setminus \{y_i\}$ for $i = 1,2,\ldots,n$. If $[X,Y]_f$ is invertible, then the inverse of $[X,Y]_f$ is the $n \times n$ matrix $B = (b_{ij})$, where*

$$\begin{aligned} b_{ij} = {} & \frac{(-1)^{i+j}}{f(x_j) f(y_i) \det[X,Y]_f} \left(\prod_{v=1}^{n} f(x_v) f(y_v)\right) \\ & \times \left(\sum_{1\le k_1<k_2<\cdots<k_{n-1}\le m} \det E(X_j)_{(k_1,k_2,\ldots,k_{n-1})} \det E(Y_i)_{(k_1,k_2,\ldots,k_{n-1})} \right. \\ & \left. \times \Psi_{D,1/f}(d_{k_1}) \Psi_{D,1/f}(d_{k_2}) \cdots \Psi_{D,1/f}(d_{k_{n-1}})\right). \end{aligned} \tag{6.12}$$

Theorem 6.8. *Let $S_i = S \setminus \{x_i\}$ for $i = 1,2,\ldots,n$. If $[S]_f$ is invertible, then the inverse of $[S]_f$ is the $n \times n$ matrix $B = (b_{ij})$, where*

$$\begin{aligned} b_{ij} = {} & \frac{(-1)^{i+j}}{f(x_i) f(x_j) \det[S]_f} \left(\prod_{v=1}^{n} f(x_v)^2\right) \\ & \times \left(\sum_{1\le k_1<k_2<\cdots<k_{n-1}\le m} \det E(S_i)_{(k_1,k_2,\ldots,k_{n-1})} \det E(S_j)_{(k_1,k_2,\ldots,k_{n-1})} \right. \\ & \left. \times \Psi_{D,1/f}(d_{k_1}) \Psi_{D,1/f}(d_{k_2}) \cdots \Psi_{D,1/f}(d_{k_{n-1}})\right). \end{aligned} \tag{6.13}$$

Theorem 6.9. *Suppose that S is meet-closed. If $[S]_f$ is invertible, then the inverse of $[S]_f$ is the $n \times n$ matrix $B = (b_{ij})$, where*

$$b_{ij} = \frac{1}{f(x_i) f(x_j)} \sum_{x_i \vee x_j \preceq x_k} \frac{\mu_S(x_i, x_k) \mu_S(x_j, x_k)}{\Psi_{S,1/f}(x_k)}. \tag{6.14}$$

Here μ_S is the Möbius function of the poset $(S, \preceq)$.

Theorem 6.10. *Suppose that S is lower-closed. If $[S]_f$ is invertible, then the inverse of $[S]_f$ is the $n \times n$ matrix $B = (b_{ij})$, where*

$$b_{ij} = \frac{1}{f(x_i) f(x_j)} \sum_{x_i \vee x_j \preceq x_k} \frac{\mu(x_i, x_k) \mu(x_j, x_k)}{\Psi_{S,1/f}(x_k)}. \tag{6.15}$$

Here μ is the Möbius function of $(P, \preceq)$.

Corollary 6.11 [20, Theorem 2]. *Let S be a factor-closed set of distinct positive integers, and let f be a quasimultiplicative arithmetical function such that $f(r) \neq 0$ for all $r \in \mathbb{Z}^+$. If $[S]_f$ is invertible, then the inverse of $[S]_f$ is the $n \times n$ matrix $B = (b_{ij})$, where*

$$b_{ij} = \frac{1}{f(x_i) f(x_j)} \sum_{[x_i, x_j] | x_k} \frac{\mu(x_k/x_i) \mu(x_k/x_j)}{((1/f) * \mu)(x_k)}. \tag{6.16}$$

Here μ is the number-theoretic Möbius function.

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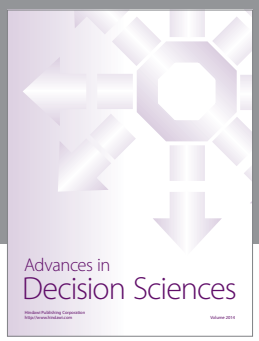
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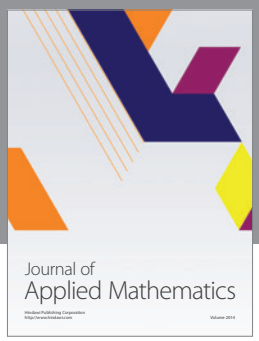
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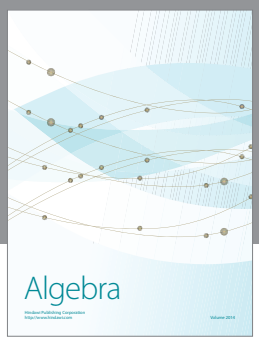
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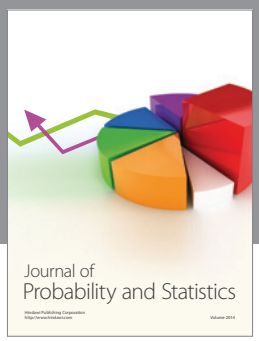
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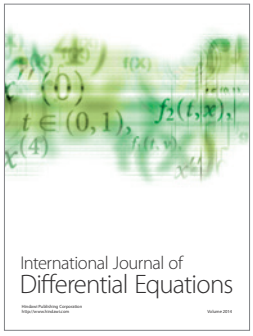


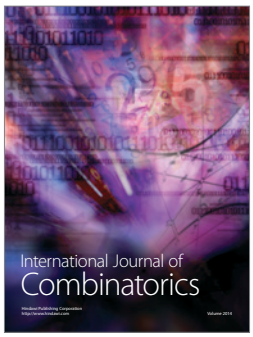


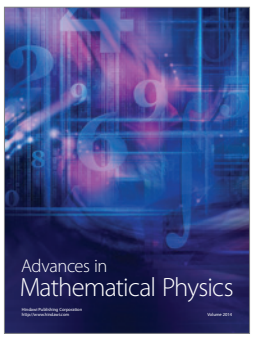


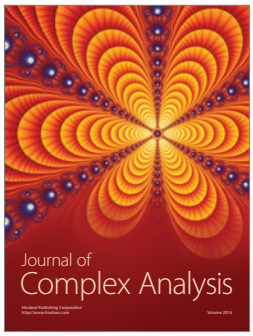




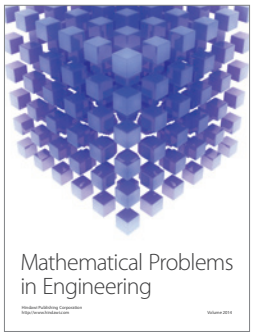


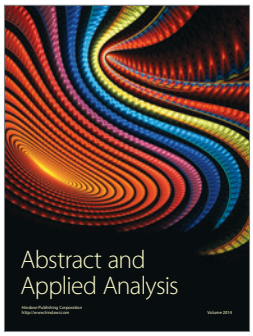


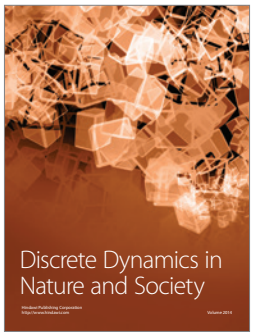


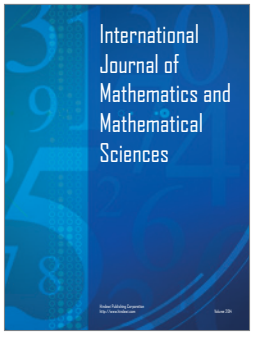






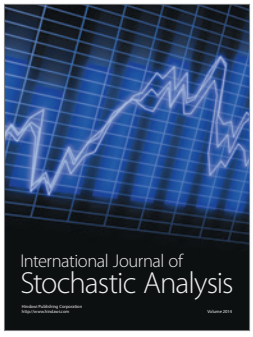














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