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Spectral properties of Sturm–Liouville system with eigenvalue-dependent boundary conditions

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In this paper, we consider the boundary value problem

$$y_j'' + \lambda^2 y_j = \sum_{k=1}^{n} V_{jk}(x) y_k, \quad x \in \mathbb{R}_+ := (0, \infty),$$

$$y_j'(0) + (\alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2) y_j(0) = 0, \quad j = 1, 2, \ldots, n,$$

where λ is the spectral parameter and $V(x) = \|V_{jk}(x)\|_1^n$ is a Hermitian matrix such that

$$\int_x^\infty t|V(t)|dt < \infty, \quad x \in \mathbb{R}_+$$

and $\alpha_i \in \mathbb{C}, i = 0, 1, 2$, with $\alpha_2 \neq 0$. In this paper, we investigate the eigenvalues and spectral singularities of L. In particular, we prove that L has a finite number of eigenvalues and spectral singularities with finite multiplicities, under the Naimark and Pavlov conditions.

Keywords: Spectrum; eigenvalue; eigenfunction; resolvent.

Mathematics Subject Classification: 34L40, 34L05, 47E05

1. Introduction

The spectral analysis of non-self-adjoint differential operators with continuous and discrete spectrum was investigated by Naimark [13]. He showed the existence of spectral singularities (which are called spectral singularities by Schwartz [15]) in the continuous spectrum of the non-self-adjoint differential operator L_0, generated in $L_2(\mathbb{R}_+)$ by the differential expression

$$l_o(y) = -y'' + q(x)y, \quad x \in \mathbb{R}_+ = (0, \infty),$$

with the boundary condition $y'(0) - hy(0) = 0$, where q is a complex valued function and $h \in \mathbb{C}$. If the condition

$$\int e^{\varepsilon x}|q(x)|dx < \infty, \quad \varepsilon > 0,$$

is satisfied, then L_0 has a finite number of eigenvalues and spectral singularities with finite multiplicities.

Pavlov [14] proved that if

$$\sup_{x\in\mathbb{R}_+}[e^{\varepsilon\sqrt{x}}|q(x)|] < \infty, \quad \varepsilon > 0,$$

then the eigenvalues and the spectral singularities are of finite number and each eigenvalue is of finite multiplicity. Lyance [10] has studied the effect of spectral singularities in the spectral expansion in terms of the principal functions for the operator L_0. The spectral singularities of the discrete Schrödinger, Dirac and infinite Jacobi matrices have been investigated in [1, 2, 11, 3].

Binding *et al.* [5–7], Code [8] and Bairamov and Yokus [4] have studied the spectral theory of regular Sturm–Liouville operators with boundary conditions depending on the spectral parameter.

Let L denote the operator generated in $L_2(\mathbb{R}_+)$ by

$$y_j'' + \lambda^2 y_j = \sum_{k=1}^{n} V_{jk}(x)y_k, \quad x \in \mathbb{R}_+ := (0,\infty), \tag{1}$$

$$y_j'(0) + (a_0 + \alpha_1\lambda + \alpha_2\lambda^2)y_j(0) = 0, \quad j = 1, 2, \ldots, n, \tag{2}$$

where λ is the spectral parameter and $V(x) = \|V_{jk}(x)\|_1^n$ is a Hermitian matrix such that

$$\int_x^\infty t|V(t)|dt < \infty, \quad x \in \mathbb{R}_+ \tag{3}$$

and $\alpha_i \in \mathbb{C}, i = 0, 1, 2$, with $\alpha_2 \neq 0$.

In this paper, we investigate the eigenvalues and the spectral singularities of L. In particular, we show that L has a finite number of eigenvalues and spectral singularities with finite multiplicities under the Naimark and Pavlov conditions:

$$V \in \mathrm{AC}(\mathbb{R}_+), \quad \lim_{x\to\infty} V(x) = 0, \quad \int_0^\infty e^{\varepsilon x}|V'(x)|dx < \infty, \quad \varepsilon > 0,$$

$$V \in \mathrm{AC}(\mathbb{R}_+), \quad \lim_{x\to\infty} V(x) = 0, \quad \sup_{x\in\mathbb{R}_+}[e^{\varepsilon\sqrt{x}}|V'(x)|] < \infty, \quad \varepsilon > 0,$$

respectively, where $\mathrm{AC}(\mathbb{R}_+)$ denotes the class of complex-valued absolutely continuous function on $\mathbb{R}_+$.

2. Jost Functions of (1)–(2)

Under the condition

$$\int_x^\infty x|V(x)|dx < \infty, \quad x \in \mathbb{R}_+.$$

Eq. (1) has a solution $E(x,\lambda)$ satisfying

$$\lim_{x\to\infty} E(x,\lambda)e^{-i\lambda x} = I, \quad \lambda \in \overline{\mathbb{C}}_+,$$

where $\overline{\mathbb{C}}_+ = \{\lambda : \lambda \in \mathbb{C}, \operatorname{Im}\lambda \geq 0\}$. The solution $E(x,\lambda)$ is called Jost solution of (1). Note that Jost solution has a representation [12]:

$$E(x,\lambda) = e^{i\lambda x}I + \int_x^\infty K(x,t)e^{i\lambda t}dt, \quad \lambda \in \overline{\mathbb{C}}_+, \tag{4}$$

where I is identity matrix and the matrix $K(x,t)$ is the solution of the integral equation

$$\begin{aligned} K(x,t) = {} & \frac{1}{2}\int_{\frac{x+t}{2}}^\infty V(s)ds + \frac{1}{2}\int_x^{\frac{(x+t)}{2}} \int_{t+x-s}^{t+s-x} V(s)K(s,u)duds \\ & + \frac{1}{2}\int_{\frac{x+t}{2}}^\infty \int_s^{t+s-x} V(s)K(s,u)duds \quad (0 < x \leq t) \end{aligned} \tag{5}$$

and $K(x,t)$ is continuously differentiable with respect to its arguments. We also have

$$\begin{aligned} |K(x,t)| &\leq c\sigma\left(\frac{x+t}{2}\right), \\ |K_x(x,t)|, \quad |K_t(x,t)| &\leq \frac{1}{4}\left|V\left(\frac{x+t}{2}\right)\right| + c\sigma\left(\frac{x+t}{2}\right), \end{aligned} \tag{6}$$

where $\sigma(x) = \int_x^\infty |V(s)|ds$ and $c > 0$ is a constant.

Let

$$F^+(\lambda) := E'(0,\lambda) + (\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2)E(0,\lambda), \quad \lambda \in \overline{\mathbb{C}}_+, \tag{7}$$

$$F^-(\lambda) := E'(0,-\lambda) + (\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2)E(0,-\lambda), \quad \lambda \in \overline{\mathbb{C}}_-, \tag{8}$$

where F^+ and F^- are analytic in $\overline{\mathbb{C}}_+ = \{\lambda : \lambda \in \mathbb{C}, \operatorname{Im}\lambda \geq 0\}$, and $\overline{\mathbb{C}}_- = \{\lambda : \lambda \in \mathbb{C}, \operatorname{Im}\lambda \leq 0\}$, respectively, and continuous up to the real axis. The functions F^+ and F^- are called Jost functions of L.

Let us denote the eigenvalues and the spectral singularities of L by $\sigma_d(L)$ and $\sigma_{ss}(L)$, respectively. It is evident that

$$\sigma_d(L) = \{\lambda : \lambda \in \mathbb{C}_+, \det F^+(\lambda) = 0\} \cup \{\lambda : \lambda \in \mathbb{C}_-, \det F^-(\lambda) = 0\}, \tag{9}$$

$$\sigma_{ss}(L) = \{\lambda : \lambda \in \mathbb{R}^*, \det F^+(\lambda) = 0\} \cup \{\lambda : \lambda \in \mathbb{R}^*, \det F^-(\lambda) = 0\}, \tag{10}$$

where $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$.

Definition 2.1. The multiplicity of a zero $\det F^\pm(\lambda)$ in $\overline{\mathbb{C}}_\pm$ is defined as the multiplicity of the corresponding eigenvalue and spectral singularity of L.

To investigate the properties of the eigenvalues and the spectral singularities of L, we need to discuss the properties of the zeros of $\det F^\pm(\lambda)$ in $\overline{\mathbb{C}}_\pm$.

Define

$$\begin{aligned} M_1^{\pm} &= \{\lambda : \lambda \in \mathbb{C}_{\pm}, \det F^{\pm}(\lambda) = 0\}, \\ M_2^{\pm} &= \{\lambda : \lambda \in \mathbb{R}^*, \det F^{\pm}(\lambda) = 0\}, \end{aligned} \tag{11}$$

then by (9), (10) we have

$$\begin{aligned} \sigma_d(L) &= M_1^+ \cup M_1^-, \\ \sigma_{ss}(L) &= M_2^+ \cup M_2^-. \end{aligned} \tag{12}$$

Now let us assume that

$$V \in \mathrm{AC}(\mathbb{R}_+), \quad \lim_{x\to\infty} V(x) = 0, \quad \int_0^\infty x^3 |V'(x)| dx < \infty. \tag{13}$$

Theorem 2.2. *Under condition* (13), *the functions* $F^{\pm}(\lambda)$ *have the representations*

$$F^+(\lambda) = (\alpha_2\lambda^2 + \beta^+\lambda + \delta^+)I + \int_0^\infty G^+(t)e^{i\lambda t}\, dt, \quad \lambda \in \overline{\mathbb{C}}_+, \tag{14}$$

$$F^-(\lambda) = (\alpha_2\lambda^2 + \beta^-\lambda + \delta^-)I + \int_0^\infty G^-(t)e^{i\lambda t}\, dt, \quad \lambda \in \overline{\mathbb{C}}_-, \tag{15}$$

where $\beta^{\pm}, \delta^{\pm} \in \mathbb{C}$ *and* $G^{\pm} \in L_1(\mathbb{R}_+)$.

Proof. Using (4)–(6), we get:

$$\begin{aligned} F^+(\lambda) = &\left[i\lambda + \int_0^\infty K_x(0,t)e^{i\lambda t}\, dt - K(0,0)\right] I + (\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2) \\ &\times \left(I + \int_0^\infty K(0,t)e^{i\lambda t}\, dt\right). \end{aligned} \tag{16}$$

By integration by parts, we get:

$$\begin{aligned} F^+(\lambda) = &\alpha_2\lambda^2 I + \lambda\left(\alpha_1 + i\alpha_2 K(0,0) + i\right) - K(0,0) \\ &+ i\alpha_1 K(0,0) - \alpha_2 K_t(0,0) + \alpha_0 I \\ &+ \int_0^\infty \left[K_x(0,t) + \alpha_0 K(0,t) + i\alpha_1 K_t(0,t) - \alpha_2 K_{tt}(0,t)\right] e^{i\lambda t}\, dt. \end{aligned} \tag{17}$$

From (17) we get (14) where

$$\begin{aligned} \beta^+ &= \alpha_1 + i\alpha_2 K(0,0) + i, \\ \delta^+ &= -K(0,0) + i\alpha_1 K(0,0) - \alpha_2 K_t(0,0) + \alpha_0 I, \\ G^+(t) &= K_x(0,t) + \alpha_0 K(0,t) + i\alpha_1 K_t(0,t) - \alpha_2 K_{tt}(0,t). \end{aligned} \tag{18}$$

From (5), we compute:

$$\begin{aligned} K_{tt}(0,t) = &-\frac{1}{8}V'\left(\frac{t}{2}\right) + \frac{1}{2}\int_0^\infty V(s)K_t(s,t+s)ds - \frac{1}{2}\int_0^{\frac{t}{2}} V(s)K_t(s,t-s)ds \\ &- \frac{1}{4}V\left(\frac{t}{2}\right) K\left(\frac{t}{2},\frac{t}{2}\right). \end{aligned} \tag{19}$$

From (19), we see that

$$|K_{tt}(0,t)| \le c\left[t\left|V\left(\frac{t}{2}\right)\right| + \left|V'\left(\frac{t}{2}\right)\right| + t\sigma\left(\frac{t}{2}\right) + \sigma_1\left(\frac{t}{2}\right)\right], \tag{20}$$

holds, where $\sigma_1(t) = \int_t^\infty \sigma(s)ds$ and $c > 0$ is a constant.

It follows from (6) and (20) that $K(x,\cdot),\ K_x(x,\cdot),\ K_t(x,\cdot),\ K_{tt}(x,\cdot) \in L_1(\mathbb{R}_+)$. It is clear that $G^+ \in L_1(\mathbb{R}_+)$ from (18). In a similar way we obtain (15). □

Theorem 2.3. *Under condition* (13)*, we have the following*:

(i) *The set of $\sigma_d(L)$ is bounded and has at most countable number of elements, and its limit points can lie only in a bounded subinterval of the real axis.*
(ii) *The set of $\sigma_{ss}(L)$ is bounded and its linear Lebesgue measure is zero.*

Proof. From (15), (17) and (20), we see that

$$F^+(\lambda) = (\alpha_2\lambda^2 + \beta^+\lambda + \delta^+)I + o(1), \quad \lambda \in \overline{\mathbb{C}}_+,\ |\lambda| \to \infty, \tag{21}$$

$$F^-(\lambda) = (\alpha_2\lambda^2 + \beta^-\lambda + \delta^-)I + o(1), \quad \lambda \in \overline{\mathbb{C}}_-,\ |\lambda| \to \infty. \tag{22}$$

Using (9), (10), (21), (22) and uniqueness theorem of analytic functions [9], we get both parts of Theorem 2.3. □

3. Naimark and Pavlov Conditions for L

We will denote the set of all limit points of $\sigma_d(L)$ by M_3^+ and M_3^-, respectively, and the set of all zeros of $\det F^+(\lambda)$ and $\det F^-(\lambda)$ with infinity multiplicity in $\overline{\mathbb{C}}_+$ and $\overline{\mathbb{C}}_-$, by M_4^+ and M_4^-, respectively.

It is obvious that

$$M_3^\pm \subset M_2^\pm, \quad M_4^\pm \subset M_2^\pm, \quad M_3^\pm \subset M_4^\pm, \tag{23}$$

and the linear Lebesgue measures of $M_3^\pm$ and $M_4^\pm$ are zero.

Lemma 3.1. *If*

$$V \in \mathrm{AC}(\mathbb{R}_+), \quad \lim_{x\to\infty} V(x) = 0, \quad \int_0^\infty e^{\varepsilon x}|V'(x)|dx < \infty, \quad \varepsilon > 0, \tag{24}$$

then the operator L has a finite number of eigenvalues and spectral singularities, and each of them is of a finite multiplicity.

Proof. From (18), we have:

$$\int_0^\infty |G^+(t)e^{i\lambda t}\,dt| \le \int_0^\infty |K_x(0,t)e^{-t\,\mathrm{Im}\,\lambda}|dt + |\alpha_0|\int_0^\infty |K(0,t)e^{-t\,\mathrm{Im}\,\lambda}|dt$$
$$+ |\alpha_1|\int_0^\infty |K_t(0,t)e^{-t\,\mathrm{Im}\,\lambda}|dt + |\alpha_2|\int_0^\infty |K_{tt}(0,t)e^{-t\,\mathrm{Im}\,\lambda}|dt. \tag{25}$$

It can be shown by (24) that four integrals on the right-hand side of (25) are uniformly convergent for λ if and only if $\operatorname{Im}\lambda > -\frac{\varepsilon}{4}$. Hence, by (25), we observe that the function $F^+(\lambda)$ has an analytic continuation to the half-plane $\operatorname{Im}\lambda > -\frac{\varepsilon}{4}$. Hence $\det F^+(\lambda)$ cannot have any zero with infinite multiplicity in $\overline{\mathbb{C}}_+$ by [9]. Thus we get that $M_4^+ = \emptyset$. It follows from (23) that $M_3^+ = \emptyset$. Also, by Bolzano–Weierstrass Theorem and [9], the sets M_1^+ and M_2^+ cannot have infinitely many elements. Therefore, the sets M_1^+ and M_2^+ have a finite number of elements with a finite multiplicity. We obtain similar results for the sets M_1^- and M_2^-. By (9), (10) we have the proof of the theorem. □

Now let us assume that

$$V \in \mathrm{AC}(\mathbb{R}_+), \quad \lim_{x\to\infty} V(x) = 0, \quad \sup_{x\in\mathbb{R}_+} [e^{\varepsilon\sqrt{x}}|V'(x)|] < \infty, \quad \varepsilon > 0. \tag{26}$$

Now let us recall the Pavlov's Theorem which will be required for the proof of the next lemma.

Pavlov's Theorem [14]. *Let function f has every order with derivatives in $\overline{\mathbb{C}}_+$ and let us denote the set of all zeros of f with infinite multiplicity by E. Suppose there exists A_n such that*

$$|f^{(n)}(z)| \le A_n, \quad z \in \overline{\mathbb{C}}_+,\ n = 0, 1, \ldots$$

and

$$\int_0 \ln T(s) d\mu(E, s) = -\infty,$$

where

$$T(s) = \inf_n \frac{A_n s^n}{n!}.$$

Moreover, if for an N positive real number

$$\left| \int_{-\infty}^{N} \frac{\ln|f(x)|}{1+x^2}\, dx \right| < \infty, \quad \left| \int_{N}^{\infty} \frac{\ln|f(x)|}{1+x^2}\, dx \right| < \infty$$

then f is equal to zero identically.

Using Pavlov's Theorem the following lemma can be given.

Lemma 3.2. *It holds that $M_2^+ = \emptyset, M_2^- = \emptyset$.*

Proof. From (14) and (26), we find that the function F^+ is analytic in $\mathbb{C}_+$ and all of its derivatives are continuous in $\overline{\mathbb{C}}_+$. For a sufficiently large $T > 0$, we have

$$\left| \frac{d^k}{d\lambda^k} F^+(\lambda) \right| \le A_k, \quad \lambda \in \overline{\mathbb{C}}_+,\ |\lambda| \le T,\ k = 0, 1, \ldots, \tag{27}$$

where

$$A_k = 2^k c \int_0^\infty t^k e^{(\frac{\varepsilon}{2})\sqrt{t}} dt, \quad k = 0, 1, \ldots \tag{28}$$

and $c > 0$ is a constant. Since the function F^+ is not equal to zero identically, then by Pavlov's theorem, M_4^+ satisfies

$$\int_0^h \ln A(s) d\mu(M_{4,s}^+) > -\infty, \tag{29}$$

where $A(s) = \inf_k (A_k s^k k!), \mu(M_{4,s}^+)$ is the linear Lebesgue measure of s-neighborhood of M_4^+ [14]. Now, we obtain the following estimates for A_k:

$$A_k \leq B b^k k^k k!, \tag{30}$$

where B and b are constants depending on c and ε. From (30), we get

$$A(s) \leq \inf_k (b^k s^k k^k) \leq B \exp(-b^{-1} e^{-1} s^{-1}).$$

Now, (29) yields

$$\int_0^h \frac{1}{s} d\mu(M_{4,s}^+) < \infty. \tag{31}$$

However, (31) holds for an arbitrary h if and only if $\mu(M_{4,s}^+) = 0$ or $M_4^+ = \emptyset$. In a similar way we can prove that $M_4^- = 0$. □

Theorem 3.3. *Under condition* (26)*, the operator L has a finite number of eigenvalues and spectral singularities, and each of them is of a finite multiplicity.*

Proof. To be able to prove the theorem, we have to show that the functions F^+ and F^- have a finite number of zeros with finite multiplicities in $\overline{\mathbb{C}}_+$ and $\overline{\mathbb{C}}_-$, respectively. We give the proof for F^+.

From Lemmas 3.1 and 3.2, we find that $M_3^+ = \emptyset$. So the bounded sets M_1^+ and M_2^+ have no limit points, that is, the function F^+ has only a finite number of zeros in $\overline{\mathbb{C}}_+$. Since $M_4^+ = \emptyset$, these zeros are of finite multiplicity. □

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