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ON BANACH LATTICES WITH LEVI NORMS

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ABSTRACT. Schmidt proved that an operator T from a Banach lattice E into a Banach lattice G with property (P) is order bounded if and only if its adjoint is order bounded, and in this case T satisfies $\|\|T\|\| = \|\|T'\|\|$. In the present paper the result is generalized to Banach lattices with Levi-Fatou norm serving as range, and some characterizations of Banach lattices with a Levi norm are given. Moreover, some characterizations of Riesz spaces having property (b) are also obtained.

1. Introduction and preliminaries

The order duals $E^\sim$ and $E^{\sim\sim}$ of E are Riesz spaces whenever E is a Riesz space. It is well known that the canonical embedding $Q_E : E \to E^{\sim\sim}$ defined by $Q_E(x) = \hat{x}\ ;\ \hat{x}(f) = f(x),\ f \in E^\sim,\ x \in E$ is a lattice homomorphism. If $E^\sim$ separates the points of E, then Q_E is also one-to-one, and hence E can be considered as a Riesz subspace of $E^{\sim\sim}$. We will assume all Riesz spaces considered in this note have seperating order duals.

Recall that a Banach lattice E is said to have property (P) if there exists a positive contractive projection $P : E'' \to E$. Property (P) implies that E is Dedekind complete. Clearly, every KB-space has property (P). Also every Dedekind complete AM-space with unit has property (P). A norm on a Banach lattice is said to be a *Levi norm* if every norm bounded upward directed set of positive elements has a supremum. Obviously, each Banach lattice with a Levi norm must be Dedekind complete. A norm on a Banach lattice E is said to be a *Fatou norm* if $0 \le x_\alpha \uparrow x$ in E implies $\|x_\alpha\| \uparrow \|x\|$.

In all undefined terminology concerning Riesz spaces we will adhere to [2] and [3].

2. Some characterizations of Banach lattices with Levi norms

The following proposition generalizes Lemma 2.1 in [7] since every Banach lattice with property (P) has a Levi-Fatou norm.

Proposition 1. *Let E and F be two Banach lattices with F having a Levi norm. Then a continuous operator $T : E \to F$ is order bounded if and only if its adjoint*

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$T' : F' \to E'$ is order bounded. In particular, if F also has a Fatou norm, then T satisfies $\||T|\| = \||T'|\|$.

Proof. Let $T : E \to F$ be an order bounded operator, $f \in (F')^+$, $g \in F'$ with $|g| \leq f$. Then we have $|T'g| = |gT| \leq |g|\,|T| \leq f\,|T| = |T|'\,f$. Therefore, T' is order bounded.

For the converse, assume that T' is an order bounded operator. Hence, T'' is order bounded. Now let $x \in E^+$, $y \in E$ with $|y| \leq x$; we have $|T(y)| = |T''(y)| \leq |T''|\,(x)$. Since F has a Levi norm, there exists a function $\varphi : F'' \to F$ defined by $\varphi(x'') = \sup\{[0, |x''|] \cap F\}$ satisfying the following conditions:

(1) *For every element x'', $y'' \in F''$ with $|x''| \leq |y''|$, $0 \leq \varphi(x'') \leq \varphi(y'')$.*

(2) *For each element x of F^+, $\varphi(x) = x$.*

So, we have $|T(y)| = \varphi(|T''(y)|) \leq \varphi\,|T''|\,(x) \leq \varphi\,|T'|'\,(x)$. Hence, T is order bounded. Also, from Theorem 1.10 in [2] we see that $|T|\,x) \leq \varphi\,|T'|'\,(x)$ for all $x \in E^+$.

Now, let F have a Levi-Fatou norm, let $x'' \in F''$ and let $x_\alpha = \alpha$ for each $\alpha \in [0, |x''|] \cap F$. By the definition of φ we have $0 \leq x_\alpha \uparrow \varphi(x'')$ in F. Since the norm on F is Fatou and $\|x_\alpha\| \leq \|x''\|$ for each α, we have $\|\varphi(x'')\| \leq \|x''\|$. Therefore, $\||T|\,(x)\| \leq \left\|\varphi\,|T'|'\,(x)\right\| \leq \left\||T'|'\,(x)\right\| \leq \left\||T'|'\right\|\|x\| = \||T'|\|\,\|x\| \leq \left\||T|'\right\|\|x\| = \||T|\|\,\|x\|$. Hence, $\||T|\,(x)\| \leq \||T'|\|\,\|x\| \leq \||T|\|\,\|x\|$. This implies that $\||T|\| = \||T'|\|$. □

The preceding proposition coupled with Theorem 3.5 in [1] yields the following characterization.

Corollary 1. *Let F be a Banach lattice with Dedekind complete. Then the following statements are equivalent:*

(a) *F has a Levi norm.*

(b) *For every Banach lattice E, if $T' \in L_b(F', E')$ when $T \in L(E, F)$, then $T \in L_b(E, F)$.*

Let E be a Banach lattice and let x'' be a positive element of E''. Let $I_{x''}$ denote the principal ideal generated by x'' in E'' and let $E_{x''}$ denote the Riesz space $I_{x''} \cap E$. It is known that $E_{x''}$ is an AM-space with the norm

$$\|u\|_{x''} = \inf\{\lambda > 0 : |u| \leq \lambda x''\}, u \in E_{x''}.$$

The proof of the following proposition is straightforward and is therefore omitted.

Proposition 2. *Let E be a Banach lattice with an order continuous norm. Then E has a Levi norm (i.e., E is a KB-space) if and only if the norm $\|.\|_{x''}$ on the Banach lattice $E_{x''}$ is a Levi norm for each $x'' \in (E'')^+$.*

Let Ω be the collection of all norm bounded upward directed sets of positive elements of E. Then for each $A \in \Omega$ the formula

$$\rho_A(x') = \sup\{|x'|\,(x) : x \in A\},\ x' \in E',$$

defines a Riesz seminorm on E', and thus, Ω generates a locally convex-solid topology on E' via the family of seminorms $\{\rho_A : A \in \Omega\}$. This topology is denoted by $|\sigma_l|\,(E', E)$.

It is clear that the well-known absolute weak topology $|\sigma|\,(E', E)$ on E' is coarser than $|\sigma_l|\,(E', E)$ (i.e., $|\sigma|\,(E', E) \subseteq |\sigma_l|\,(E', E)$).

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The following proposition shows that the inclusion $|\sigma|(E', E) \subseteq |\sigma_l|(E', E)$ is proper when E does not have any Levi norm.

Proposition 3. *Let E be a Dedekind complete Banach lattice. Then E has a Levi norm if and only if $|\sigma|(E', E) = |\sigma_l|(E', E)$.*

Proof. Let E have a Levi norm. Then it is clear that $|\sigma|(E', E) = |\sigma_l|(E', E)$.

To prove the converse, suppose that the equality $|\sigma|(E', E) = |\sigma_l|(E', E)$ is satisfied and $\{x_\alpha\}$ is an increasing, norm-bounded net in E^+. Then $A = \{x_\alpha : \alpha \in \Lambda\}$ is order bounded in E''. By Theorem 11.11 in [2], there exists an element x of E^+ with $[-x, x]^o \subseteq \{x' \in E' : \rho_A(x') \leq 1\} \subseteq A^\circ$ in E'. By the Bipolar Theorem, we have $[-x, x]^{\circ\circ} = [-x, x]$. It follows that $A \subseteq [-x, x]$ and E has a Levi norm. □

3. Some characterizations of Riesz spaces with property (b)

Let E be a Riesz space and let A be a subset of E. If A is order bounded in $E^{\sim\sim}$, then A is said to be b-order bounded in E. A Riesz space E is said to have property (b) if every b-order bounded subset of E is order bounded in E [4]. Every order dual and perfect Riesz space has property (b); if $E^{\sim\sim}$ retractable on E, then E has property (b), and for an arbitrary compact Hausdorff space K, $C(K)$ has property (b). But, c_0 does not have the property (b) [4]. The Levi property and Zaanen's B-property also imply property (b) in Frechet lattices. However, $C[0, 1]$, with the supremum norm, has neither the Levi property nor the B-property [4].

A Riesz space E has property (b) if and only if for each net $\{x_\alpha\}$ in E satisfying $0 \leq x_\alpha \uparrow x''$ for some x'' in $E^{\sim\sim}$, $\{x_\alpha\}$ is order bounded in E [4].

In [5], it is seen that $\mathcal{L}_b(E, F)$ has property (b) if and only if F has property (b), where both E and F are two Banach lattices and F is Dedekind complete. The following proposition generalizes the result to the case of Riesz spaces.

Proposition 4. *If E and F are two Riesz spaces with F Dedekind complete, then $\mathcal{L}_b(E, F)$ has property (b) if and only if F has property (b).*

Proof. Assume first that $\mathcal{L}_b(E, F)$ has property (b). Take a net $\{y_\alpha\}$ in F with $0 \leq y_\alpha \uparrow y''$ in $F^{\sim\sim}$. Then we choose $f \in (E^\sim)^+$ with $f \neq 0$. Define $\phi : F \to \mathcal{L}_b(E, F)$, $y \to f \otimes y$, $(f \otimes y)(x) = f(x).y$ for each $x \in E$. It is known that ϕ is a Riesz isomorphism. Since $\{y_\alpha\}$ is b-order bounded in F, $\phi(y_\alpha)$ is b-order bounded in $\mathcal{L}_b(E, F)$. From the hypothesis there exists T in $\mathcal{L}_b(E, F)$ such that $0 \leq f \otimes y_\alpha \leq T$. Hence, there exists a positive element x_o in E such that $0 \leq y_\alpha \leq T(x_o)$ in F. Thus F has property (b).

Suppose now F has property (b) and (T_α) is a net in $\mathcal{L}_b(E, F)$ with $0 \leq T_\alpha \uparrow T^{**}$ in $[\mathcal{L}_b(E, F)]^{\sim\sim}$. We choose $x \in E^+$ with $x \neq 0$. Define $\psi : F^\sim \to \mathcal{L}_b(E, F)^\sim$, $f \to \psi(f)$, $\psi(f)(T) = f(Tx)$, $T \in \mathcal{L}_b(E, F)$. Clearly ψ is one-to-one and positive. Therefore, we have $0 \leq f(T_\alpha(x)) \leq T^{**}(\psi(f))$, for each $f \in (F^\sim)^+$. Thus, $\{T_\alpha(x)\}$ is a b-order bounded set in F for each $x \in E^+$. From the hypothesis there exists $y(x) \in F$ such that $0 \leq T_\alpha(x) \uparrow y(x)$ for each $x \in E^+$. The mapping $T : E^+ \to F^+$ defined by $T(x) = \sup T_\alpha(x)$ is additive on E^+. We can extend T to a unique positive operator on E. Clearly $0 \leq T_\alpha \leq T$, and so $\mathcal{L}_b(E, F)$ has property (b). □

Furthermore, the following question, "when does the Dedekind completion of a Riesz space have property (b)?" is taken into consideration.

Proposition 5. *Let E be a Riesz space. Then E has property (b) if and only if Dedekind completion of E has property (b).*

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Proof. Let E have property (b) and let E^δ be a Dedekind completion of E. By Theorem 7.12 in [2], E is Riesz isomorphic to a Riesz subspace of E^δ such that for each $x \in E^\delta$ we have

$$x = \sup\{y \in E : y \leq x\} = \inf\{z \in E : x \leq z\},$$

given a net $\{x_\alpha^\delta\}$ in E^δ and given $0 \leq x'' \in (E^\delta)^{\sim\sim}$ such that $0 \leq x_\alpha^\delta \uparrow\leq x''$ in $(E^\delta)^{\sim\sim}$. Thus, there exists a net $\{x_{\beta(\alpha)}\} \subset E$ such that $0 \leq x_{\beta(\alpha)} \uparrow_{\beta(\alpha)} x_\alpha^\delta \leq x''$ in $(E^\delta)^{\sim\sim}$ for each α. Therefore, we have $0 \leq f(x_{\beta(\alpha)}) \leq x''(f)$ for each $f \in ((E^\delta)^\sim)^+$. By Theorem 2.8 in [2] for each $0 \leq f \in E^\sim$, there exists an extension $\hat{f}$ from E^δ into $\mathbb{R}$. Consequently, we have $0 \leq f(x_{\beta(\alpha)}) \leq \hat{f}(x_\alpha^\delta) \leq x''(\hat{f})$. Hence, the set $\{x_{\beta(\alpha)}\}$ is b-order bounded in E. Thus, by hypothesis, there exists x in E such that $0 \leq x_{\beta(\alpha)} \leq x$ for all $\alpha, \beta(\alpha)$. So, we have $0 \leq x_\alpha^\delta \leq x$. Therefore, E^δ has property (b).

To prove the converse, suppose that E^δ has property (b). Let $\{x_\alpha\}$ be a net in E with $0 \leq x_\alpha \uparrow\leq x''$ in $E^{\sim\sim}$. Define the functional $\hat{x} : (E^\delta)^\sim \to \mathbb{R}$ by $\hat{x} : f \to x''(f_{|E})$. Then we have $0 \leq x_\alpha \uparrow\leq \hat{x}$ in $(E^\delta)^{\sim\sim}$. Thus, there exists a positive element z in E with $0 \leq x_\alpha \leq z$. Hence, E has property (b), as desired. □

Let $\{E_i : i \in I\}$ be a family of Riesz spaces. Then the cartesian product $\prod_{i\in I} E_i$, under the ordering $\{x_i\} \leq \{y_i\}$ whenever $x_i \leq y_i$ holds for all $i \in I$, is a Riesz space. The direct sum $\sum_{i\in I} \bigoplus E_i$ is the vector space of all elements $\{x_i\}$ of $\prod_{i\in I} E_i$ for which $x_i = 0$ holds for all but a finite number of i. $\sum_{i\in I} \bigoplus E_i$ is a Riesz subspace of $\prod_{i\in I} E_i$ with the pointwise algebraic and lattice operations [2]. The next proposition characterizes the direct sum $\sum_{i\in I} \bigoplus E_i$ with property (b).

Proposition 6. *Let $\{E_i : i \in I\}$ be a family of Riesz spaces. The direct sum $\sum_{i\in I} \oplus E_i$ has property (b) if and only if each E_i has property (b).*

Proof. Let E_i have property (b) for each $i \in I$, given a net $\{x_\alpha = (x_\alpha^i),\ x_\alpha^i \in E_i$ for each $i \in I,\ \alpha \in \Lambda\}$ in $\sum_{i\in I} \bigoplus E_i$ with $0 \leq x_\alpha \uparrow\leq x''$ in $(\sum_{i\in I} \bigoplus E_i)^{\sim\sim}$. It is enough to show that there exists a finite subset J of I such that $x_\alpha^i = 0$ for all α and $i \in I \setminus J$. To see this, assume by way of contradiction that there exist $\alpha_n \in \Lambda$ and $i_n \in I$ such that $x_{\alpha_n}^{i_n} \neq 0$ for each $n \in \mathbb{N}$. One can choose $\alpha_n \in \Lambda$ with $\alpha_n \leq \alpha_{n+1}$ for each $n \in \mathbb{N}$. Since $E_{i_n}^\sim$ separates the points of E_{i_n} for each i_n, then there exists $g_{i_n} \in (E_{i_n}^\sim)^+$ with $g_{i_n}(x_{\alpha_n}^{i_n}) = 1$. Now define a functional g:

$$g = (g_i) = \begin{cases} g_{i_n}, & \text{if there exists an } n \in \mathbb{N} \text{ such that } i = i_n, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $g \in \prod_{i\in I} E_i^\sim$. On the other hand, the operator $\varphi : \prod_{i\in I} E_i^\sim \to (\sum_{i\in I} \bigoplus E_i)^\sim$, defined by $\varphi(f) = \hat{f}$, $\hat{f}(x) = \sum_{i\in I} f_i(x_i)$, $f = (f_i) \in \prod_{i\in I} E_i^\sim$ and $x = (x_i) \in \sum_{i\in I} \bigoplus E_i$, is a positive operator. Therefore, $0 \leq \sum_{i\in I} g_i(x_\alpha^i) \leq x''(\varphi(g))$ for each $\alpha \in \Lambda$ and $i \in I$. Hence, we see that $n \leq \sum_{i\in I} g_i(x_\alpha^i) \leq x''(\varphi(g))$ for each n, which is impossible. Thus, $\sum_{i\in I} \bigoplus E_i$ has property (b).

The converse is routine and is omitted. □

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