Turkish Journal of Mathematics

Volume 48 | Number 2 Article 17

3-8-2024

Laguerre type twice-iterated Appell polynomials

NESLİHAN BİRİCİK
neslihanbiricik@gazi.edu.tr

MEHMET ALİ ÖZARSLAN
mehmetali.ozarslan@emu.edu.tr

BAYRAM ÇEKİM
bayramcekim@gazi.edu.tr

Follow this and additional works at: https://journals.tubitak.gov.tr/math

 Part of the Mathematics Commons

Recommended Citation

BİRİCİK, NESLİHAN; ÖZARSLAN, MEHMET ALİ; and ÇEKİM, BAYRAM (2024) "Laguerre type twice-iterated Appell polynomials," *Turkish Journal of Mathematics*: Vol. 48: No. 2, Article 17. https://doi.org/10.55730/1300-0098.3511
Available at: https://journals.tubitak.gov.tr/math/vol48/iss2/17

This Article is brought to you for free and open access by TÜBİTAK Academic Journals. It has been accepted for inclusion in Turkish Journal of Mathematics by an authorized editor of TÜBİTAK Academic Journals. For more information, please contact pinar.dundar@tubitak.gov.tr.

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2024) 48: 360 – 376
© TÜBİTAK
doi:10.55730/1300-0098.3511

Laguerre type twice-iterated Appell polynomials

Neslihan BİRİCİK[1,3], **Mehmet Ali ÖZARSLAN**[2], **Bayram ÇEKİM**[3,*]

[1]Graduate School of Natural and Applied Sciences, Gazi University, Ankara, Turkiye
[2]Department of Mathematics, Faculty of Arts and Sciences, Eastern Mediterranean University, via Mersin 10, Gazimağusa, North Cyprus, Turkiye
[3]Department of Mathematics, Faculty of Science, Gazi University, Ankara, Turkiye

Received: 04.07.2023 • **Accepted/Published Online:** 30.01.2024 • **Final Version:** 08.03.2024

Abstract: In this study, we use discrete Appell convolution to define the sequence of Laguerre type twice-iterated Appell polynomials. We obtain explicit representation, recurrence relation, determinantal representation, lowering operator, integro-partial raising operator and integro-partial differential equation. In addition, the special cases of this new family are investigated using Euler and Bernoulli numbers. We also state their corresponding characteristic properties.

Key words: Laguerre type Appell polynomials, recurrence relation, shift operators, differential equation

1. Introduction

The theory of special functions has been a subject of interest in mathematics because of its useful properties in solving many applied problems. Polynomial families are an important area of special functions theory and have an important place in the literature. It has a wide working area due to its easy-to-understand and useful definition of Appell polynomials, i.e. [1, 2, 20, 21]

$$\frac{d}{dx}A_n(x) = nA_{n-1}(x), \quad n = 0, 1, 2, \ldots.$$

Appell polynomials $A_n(x)$ can be equivalently defined by the generating function as follows:

$$a(t)\, e^{xt} = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!} \tag{1}$$

where

$$a(t) = \sum_{n=0}^{\infty} A_n \frac{t^n}{n!}, \quad A_0 \neq 0 \tag{2}$$

[1, 2, 20, 21]. Some different definitions and generalizations of Appell polynomials have shed light on the investigation of their interesting properties. Thanks to their wide range of applications, some variants of Appell polynomials have been studied especially in the last years (see [10, 11, 13, 16, 17, 22–25]). Recently, twice-

*Correspondence: bayramcekim@gazi.edu.tr
2010 *AMS Mathematics Subject Classification:* 23584

This work is licensed under a Creative Commons Attribution 4.0 International License.

iterated Appell polynomials have been defined by the following generating relation in [11]

$$a_2(t)\, a_1(t)\, e^{xt} = \sum_{n=0}^{\infty} R_n^{[2]}(x)\, \frac{t^n}{n!}, \tag{3}$$

where

$$a_1(t) = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{n!}, \quad \alpha_0 \neq 0 \tag{4}$$

and

$$a_2(t) = \sum_{n=0}^{\infty} \beta_n \frac{t^n}{n!}, \quad \beta_0 \neq 0. \tag{5}$$

Laguerre type Appell polynomials have been the focus of attention of researchers. Various generalizations of these polynomials and their relationship with other polynomials continue to be studied (see [4–6, 8, 9, 12, 14, 15, 18, 19]). In one of these studies, Dattoli and Torre [5], defined bivariate Laguerre polynomials $L_n(x,y) := L_n$, as

$$L_n = n! \sum_{s=0}^{n} \frac{(-1)^s x^s y^{n-s}}{(n-s)!\,(s!)^2}. \tag{6}$$

In addition, the generating function of L_n's is as follows [6]:

$$e^{yt} C_0(xt) = \sum_{n=0}^{\infty} L_n \frac{t^n}{n!}. \tag{7}$$

where the function $C_0(x)$ is the 0-th order Tricomi function. The definition of the n-th order Tricomi functions is given in [6]

$$C_n(x) = \sum_{u=0}^{\infty} \frac{(-1)^u x^u}{u!\,(n+u)!}, \quad n \in \mathbb{N}_0. \tag{8}$$

The two-variable Laguerre type Appell polynomials ${}_L\mathcal{A}_n(x,y)$ were defined via the following generating function were [9]

$$a(t)\, e^{yt} C_0(xt) = \sum_{n=0}^{\infty} {}_L\mathcal{A}_n(x,y)\, \frac{t^n}{n!}. \tag{9}$$

We aim to create this study by using the generalization of Appell polynomials and Laguerre type Appell polynomials. We examine some characteristics of the Laguerre type twice-iterated Appell polynomials that have potential applications in various fields of applied science.

The research is structured as follows: Laguerre type twice-iterated Appell polynomials are introduced in Section 2. The equivalence theorem, the circular theorem and determinantal representation are presented. Also, for the sequence of Laguerre type twice-iterated Appell polynomials, we also derive a recurrence relation, lowering operator, integro-partial raising operator and integro-partial differential equation. For the particular examples of this family of polynomials, we introduce Laguerre type twice-iterated Euler polynomials, Laguerre type twice-iterated Bernoulli polynomials, and Laguerre type twice-iterated Bernoulli-Euler polynomials in Section 3.

2. Some characteristics of Laguerre type twice-iterated Appell polynomials

The sequence of Laguerre type twice-iterated Appell polynomials, which we have examined with ${}_{L}\mathcal{A}_{m}^{[2]}(x,y)$ in this section, has been studied. For this family of polynomials, we provide the equivalence theorem, determinantal representation and the circular theorem. For the Laguerre type twice-iterated Appell polynomials, we obtain the recurrence relation, lowering operator, integro-partial raising operator and integro-partial differential equation.

Let $Q_m(x,y)$ be the sequence of Laguerre type Appell polynomials in [9] provided by

$$Q_m(x,y)=\sum_{j=0}^{m}\sum_{l=0}^{j}\binom{m}{j}\binom{j}{l}q_{m-j}\frac{(-1)^l x^l y^{j-l}}{l!} \tag{10}$$

with

$$a_1(t)=\sum_{m=0}^{\infty}q_m\frac{t^m}{m!},\quad q_0\neq 0. \tag{11}$$

Now we introduce the Laguerre type twice-iterated Appell polynomials as follows:

Definition 1 *Let $Q_j(x,y)$ be a sequence of Laguerre type Appell polynomials given in (10) and $\mathcal{K}_m(x,y)$ be any polynomial explicitly given as follows:*

$$\mathcal{K}_m(x,y)=\sum_{j=0}^{m}a(m,j)\sum_{l=0}^{j}\binom{j}{l}\frac{(-1)^l x^l y^{j-l}}{l!}. \tag{12}$$

Let ${}_{L}\mathcal{A}_{m}^{[2]}(x,y)$ be the polynomial convolution defined by using the Laguerre type Appell polynomials as

$${}_{L}\mathcal{A}_{m}^{[2]}(x,y):=\sum_{j=0}^{m}a(m,j)\,Q_j(x,y),\quad m\in\mathbb{N}_0. \tag{13}$$

The polynomials ${}_{L}\mathcal{A}_{m}^{[2]}(x,y)$ will be called Laguerre type twice-iterated Appell polynomials, if they satisfy the relation

$$D_x\,{}_{L}\mathcal{A}_{m}^{[2]}(x,y)=m\,{}_{L}\mathcal{A}_{m-1}^{[2]}(x,y). \tag{14}$$

Theorem 1 *The statements that follow are equivalent:*

(a) The sequence of Laguerre type twice-iterated Appell polynomials is $\left\{{}_{L}\mathcal{A}_{m}^{[2]}(x,y)\right\}_{m\in\mathbb{N}}$.

(b) $\{\mathcal{K}_m(x,y)\}_{m\in\mathbb{N}}$ is the sequence of Laguerre type Appell polynomials that satisfy given by

$$a(m,j)=\binom{m}{j}a_{m-j} \tag{15}$$

and

$$a_2(t)=\sum_{m=0}^{\infty}a_m\frac{t^m}{m!}. \tag{16}$$

(c) The sequence of Laguerre type twice-iterated Appell polynomials $\left\{ {}_{L}\mathcal{A}_{m}^{[2]}(x,y)\right\}_{m\in\mathbb{N}}$ *has the following explicit representation:*

$$ {}_{L}\mathcal{A}_{m}^{[2]}(x,y)=\sum_{j=0}^{m}\sum_{l=0}^{j}\sum_{r=0}^{l}\binom{m}{j}\binom{j}{l}\binom{l}{r}a_{m-j}q_{j-l}\frac{(-1)^{r}x^{r}y^{l-r}}{r!}. \tag{17}$$

(d) The generating relation of the $\left\{ {}_{L}\mathcal{A}_{m}^{[2]}(x,y)\right\}_{m\in\mathbb{N}}$ *is provided by*

$$a_{2}(t)\,a_{1}(t)\,e^{yt}C_{0}(xt)=\sum_{m=0}^{\infty}{}_{L}\mathcal{A}_{m}^{[2]}(x,y)\,\frac{t^{m}}{m!}. \tag{18}$$

Proof $(a)\Leftrightarrow(b)$ Let $\left\{ {}_{L}\mathcal{A}_{m}^{[2]}(x,y)\right\}_{m\in\mathbb{N}}$ be a sequence of Laguerre type twice-iterated Appell polynomials. Applying the derivative operator D_x on both sides of (13), we get

$$m\,{}_{L}\mathcal{A}_{m-1}^{[2]}(x,y)=\sum_{j=1}^{m}a(m,j)\,jQ_{j-1}(x,y)=\sum_{j=0}^{m-1}a(m,j+1)\,(j+1)\,Q_{j}(x,y).$$

Therefore, we have

$$ {}_{L}\mathcal{A}_{m-1}^{[2]}(x,y)=\frac{1}{m}\sum_{j=0}^{m-1}a(m,j+1)\,(j+1)\,Q_{j}(x,y)$$

or equivalently

$$ {}_{L}\mathcal{A}_{m}^{[2]}(x,y)=\frac{1}{m+1}\sum_{j=0}^{m}a(m+1,j+1)\,(j+1)\,Q_{j}(x,y). \tag{19}$$

Comparing (13) and (19), we get

$$a(m,j)=\frac{j+1}{m+1}a(m+1,j+1).$$

When we iterate the last equation j times, we obtain

$$\begin{aligned} a(m+1,j+1) &= \frac{(m+1)\,m\ldots(m-j+1)}{(j+1)\,j\ldots1}a(m-j,0)\\ &= \binom{m+1}{j+1}a(m-j,0). \end{aligned}$$

Thus, we get

$$a(m,j)=\binom{m}{j}a_{m-j}. \tag{20}$$

When we substitute (20) in (12), we find that

$$\mathcal{K}_{m}(x,y)=\sum_{j=0}^{m}\sum_{l=0}^{j}\binom{m}{j}\binom{j}{l}a_{m-j}\frac{(-1)^{l}x^{l}y^{j-l}}{l!}. \tag{21}$$

Using the differential operator D_x on both sides of the last equation, we have

$$D_x\left(\mathcal{K}_m\left(x,y\right)\right)=m\mathcal{K}_{m-1}\left(x,y\right),\quad m\in\mathbb{N}.$$

Therefore, $\left\{\mathcal{K}_m\left(x,y\right)\right\}_{m\in\mathbb{N}}$ is a sequence of Laguerre type Appell polynomials the converse proposition $(b)\Leftrightarrow(a)$ is clearly demonstrated to be correct.

To obtain $(b)\Leftrightarrow(c)$, let $\left\{\mathcal{K}_m\left(x,y\right)\right\}_{m\in\mathbb{N}}$ be a sequence of Laguerre type Appell polynomials. By the definition of the Laguerre type twice-iterated Appell polynomials, we get

$${}_L\mathcal{A}_m^{[2]}\left(x,y\right):=\sum_{j=0}^{m}\binom{m}{j}a_{m-j}Q_j\left(x,y\right),\ m\in\mathbb{N}_0 \tag{22}$$

where

$$Q_j\left(x,y\right)=\sum_{l=0}^{j}\sum_{r=0}^{l}\binom{j}{l}\binom{l}{r}q_{j-l}\frac{\left(-1\right)^r x^r y^{l-r}}{r!}. \tag{23}$$

Hence, the explicit representation for Laguerre type twice-iterated Appell polynomials is as follows:

$$\begin{aligned}{}_L\mathcal{A}_m^{[2]}\left(x,y\right) &= \sum_{j=0}^{m}\binom{m}{j}a_{m-j}\sum_{l=0}^{j}\sum_{r=0}^{l}\binom{j}{l}\binom{l}{r}q_{j-l}\frac{\left(-1\right)^r x^r y^{l-r}}{r!}\\ &= \sum_{j=0}^{m}\sum_{l=0}^{j}\sum_{r=0}^{l}\binom{m}{j}\binom{j}{l}\binom{l}{r}a_{m-j}q_{j-l}\frac{\left(-1\right)^r x^r y^{l-r}}{r!}.\end{aligned}$$

The converse proposition $(c)\Leftrightarrow(b)$ is clearly demonstrated to be correct. Lastly, to obtain $(c)\Leftrightarrow(d)$, writing the explicit representation of ${}_L\mathcal{A}_m^{[2]}\left(x,y\right)$ and using the Cauchy product for the series, we get

$$\sum_{m=0}^{\infty}{}_L\mathcal{A}_m^{[2]}\left(x,y\right)\frac{t^m}{m!}=a_2\left(t\right)a_1\left(t\right)e^{yt}C_0\left(xt\right).$$

Thus, the proof is completed.

Theorem 2 *The sequence of Laguerre type twice-iterated Appell polynomials has the following determinantal representation:*

$${}_L\mathcal{A}_m^{[2]}=\frac{\left(-1\right)^m}{\left(\delta_0\right)^{m+1}}\begin{vmatrix} Q_0 & Q_1 & \cdots & Q_{m-1} & Q_m\\ \delta_0 & \delta_1 & \cdots & \delta_{m-1} & \delta_m\\ 0 & \delta_0 & \cdots & \binom{m-1}{1}\delta_{m-2} & \binom{m}{1}\delta_{m-1}\\ 0 & 0 & \cdots & \binom{m-1}{2}\delta_{m-3} & \binom{m}{2}\delta_{m-2}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & \delta_0 & \binom{m}{m-1}\delta_1 \end{vmatrix} \tag{24}$$

where $\delta_0,\delta_1,\delta_2,\ldots,\delta_m$ *are the coefficients of the Maclaurin series of the function* $\frac{1}{a_2(t)}$ *and* $Q_m=Q_m\left(x,y\right)$ *in* (10).

Proof Using the series representation of $\frac{1}{a_2(t)}$ as follows:

$$[a_2(t)]^{-1} = \sum_{j=0}^{\infty} \delta_j \frac{t^j}{j!},$$

applying the generating function (18), we get

$$a_1(t)\, e^{yt} C_0(xt) = \left(\sum_{j=0}^{\infty} \delta_j \frac{t^j}{j!} \right) \left(\sum_{m=0}^{\infty} {}_L\mathcal{A}_m^{[2]}(x,y) \frac{t^m}{m!} \right).$$

Hence,

$$\sum_{m=0}^{\infty} Q_m(x,y) \frac{t^m}{m!} = \left(\sum_{j=0}^{\infty} \delta_j \frac{t^j}{j!} \right) \left(\sum_{m=0}^{\infty} {}_L\mathcal{A}_m^{[2]}(x,y) \frac{t^m}{m!} \right).$$

Applying the Cauchy product, we have

$$\sum_{m=0}^{\infty} Q_m(x,y) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \binom{m}{j} \delta_j \, {}_L\mathcal{A}_{m-j}^{[2]}(x,y) \frac{t^m}{m!}.$$

By comparing the coefficients of $\frac{t^m}{m!}$ from the polynomial equation, we get

$$Q_m = \sum_{j=0}^{m} \binom{m}{j} \delta_j \, {}_L\mathcal{A}_{m-j}^{[2]}, \quad m \in \mathbb{N}_0.$$

As a result, we have the system of equations:

$$\begin{aligned}
Q_0 &= \delta_0 \, {}_L\mathcal{A}_0^{[2]}, \\
Q_1 &= \delta_0 \, {}_L\mathcal{A}_1^{[2]} + \delta_1 \, {}_L\mathcal{A}_0^{[2]}, \\
Q_2 &= \delta_0 \, {}_L\mathcal{A}_2^{[2]} + \binom{2}{1} \delta_1 \, {}_L\mathcal{A}_1^{[2]} + \delta_2 \, {}_L\mathcal{A}_0^{[2]}, \\
&\vdots \\
Q_{m-1} &= \delta_0 \, {}_L\mathcal{A}_{m-1}^{[2]} + \binom{m-1}{1} \delta_1 \, {}_L\mathcal{A}_{m-2}^{[2]} + \ldots \delta_{m-1} \, {}_L\mathcal{A}_0^{[2]}, \\
Q_m &= \delta_0 \, {}_L\mathcal{A}_m^{[2]} + \binom{m}{1} \delta_1 \, {}_L\mathcal{A}_{m-1}^{[2]} + \ldots + \delta_m \, {}_L\mathcal{A}_0^{[2]}.
\end{aligned}$$

Applying Cramer's rule, we get

$$ {}_{L}\mathcal{A}_{n}^{[2]}(x,y)=\frac{\begin{vmatrix} \delta_0 & 0 & \cdots & 0 & Q_0 \\ \delta_1 & \delta_0 & \cdots & 0 & Q_1 \\ \delta_2 & \binom{2}{1}\delta_1 & \cdots & 0 & Q_2 \\ \delta_3 & \binom{3}{2}\delta_2 & \cdots & 0 & Q_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \delta_{m-1} & \binom{m-1}{1}\delta_{m-2} & \cdots & \delta_0 & Q_{m-1} \\ \delta_m & \binom{m}{1}\delta_{m-1} & \cdots & \binom{m}{m-1}\delta_1 & Q_m \end{vmatrix}}{\begin{vmatrix} \delta_0 & 0 & \cdots & 0 & 0 \\ \delta_1 & \delta_0 & \cdots & 0 & 0 \\ \delta_2 & \binom{2}{1}\delta_1 & \cdots & 0 & 0 \\ \delta_3 & \binom{3}{2}\delta_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \delta_{m-1} & \binom{m-1}{1}\delta_{m-2} & \cdots & \delta_0 & 0 \\ \delta_m & \binom{m}{1}\delta_{m-1} & \cdots & \binom{m}{m-1}\delta_1 & \delta_0 \end{vmatrix}}. $$

By taking the transpose in the last equation, we have

$$ {}_{L}\mathcal{A}_{m}^{[2]}(x,y)=\frac{1}{(\delta_0)^{m+1}}\begin{vmatrix} \delta_0 & \delta_1 & \cdots & \delta_{m-1} & \delta_m \\ 0 & \delta_0 & \cdots & \binom{m-1}{1}\delta_{m-2} & \binom{m}{1}\delta_{m-1} \\ 0 & 0 & \cdots & \binom{m-1}{2}\delta_{m-3} & \binom{m}{2}\delta_{m-2} \\ 0 & 0 & \cdots & \binom{m-1}{3}\delta_{m-4} & \binom{m}{3}\delta_{m-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \delta_0 & \binom{m}{m-1}\delta_1 \\ Q_0 & Q_1 & \cdots & Q_{m-1} & Q_m \end{vmatrix}. $$

As a result, the proof is completed using the elementary row operations.

Theorem 3 *The following propositions are equivalent for the sequence of Laguerre type twice-iterated Appell polynomials:*

(*a*) $\left\{ {}_{L}\mathcal{A}_{m}^{[2]}(x,y)\right\}_{m\in\mathbb{N}}$ *is a Laguerre type twice-iterated Appell polynomials.*

(*b*) *The Laguerre type twice-iterated Appell polynomials* ${}_{L}\mathcal{A}_{m}^{[2]}(x,y)$ *has the generating function given by*

$$ a_2(t)\,a_1(t)\,e^{yt}C_0(xt)=\sum_{m=0}^{\infty}{}_{L}\mathcal{A}_{m}^{[2]}(x,y)\,\frac{t^m}{m!}. $$

(*c*) *The Laguerre type twice-iterated Appell polynomials* ${}_{L}\mathcal{A}_{m}^{[2]}(x,y)$ *have the determinantal representation in* (24).

Proof The proof of this theorem is clearly seen using Theorems 1 and 2.

Theorem 4 *Laguerre type twice-iterated Appell polynomials are given a recurrence relation that is satisfied by*

$$\begin{aligned} {}_{L}\mathcal{A}_{m+1}^{[2]} &= \sum_{j=0}^{m}\binom{m}{j}\phi_j\,{}_{L}\mathcal{A}_{m-j}^{[2]}+\sum_{j=0}^{m}\binom{m}{j}\rho_j\,{}_{L}\mathcal{A}_{m-j}^{[2]} \\ &+ y\,{}_{L}\mathcal{A}_{m}^{[2]}-D_x^{-1}\,{}_{L}\mathcal{A}_{m}^{[2]} \end{aligned} \tag{25}$$

where

$$\frac{a_1^{'}(t)}{a_1(t)}=\sum_{j=0}^{\infty}\phi_j\frac{t^j}{j!},\quad \frac{a_2^{'}(t)}{a_2(t)}=\sum_{j=0}^{\infty}\rho_j\frac{t^j}{j!}\quad \textit{and}\quad D_x^{-1}\ \textit{is the inverse of}\ D_x. \tag{26}$$

Proof If we take the derivative with respect to t on both sides of (18), we get

$$\begin{aligned} \sum_{m=0}^{\infty}{}_{L}\mathcal{A}_{m+1}^{[2]}\frac{t^m}{m!}= &\quad \frac{a_1^{'}(t)}{a_1(t)}a_1(t)\,a_2(t)\,e^{yt}C_0(xt) \\ &+\frac{a_2^{'}(t)}{a_2(t)}a_1(t)\,a_2(t)\,e^{yt}C_0(xt) \\ &+ye^{yt}a_1(t)\,a_2(t)\,C_0(xt) \\ &+\sum_{m=0}^{\infty}\frac{(-1)^{m+1}(m+1)\,x^{m+1}t^m}{[(m+1)!]^2}a_1(t)\,a_2(t)\,e^{yt}. \end{aligned} \tag{27}$$

Using (26), we have

$$\begin{aligned} \sum_{m=0}^{\infty}{}_{L}\mathcal{A}_{m+1}^{[2]}\frac{t^m}{m!} &= \left(\sum_{j=0}^{\infty}\phi_j\frac{t^j}{j!}\right)\left(\sum_{m=0}^{\infty}{}_{L}\mathcal{A}_{m}^{[2]}\frac{t^m}{m!}\right) \\ &+\left(\sum_{j=0}^{\infty}\rho_j\frac{t^j}{j!}\right)\left(\sum_{m=0}^{\infty}{}_{L}\mathcal{A}_{m}^{[2]}\frac{t^m}{m!}\right)+y\sum_{m=0}^{\infty}{}_{L}\mathcal{A}_{m}^{[2]}\frac{t^m}{m!} \\ &+\left(\sum_{m=0}^{\infty}\frac{(-1)^{m+1}(m+1)\,x^{m+1}t^m}{[(m+1)!]^2}\right)a_1(t)\,a_2(t)\,e^{yt}. \end{aligned}$$

Hence, when use the Cauchy product, we obtain

$$\begin{aligned} \sum_{m=0}^{\infty}{}_{L}\mathcal{A}_{m+1}^{[2]}\frac{t^m}{m!} &= \sum_{m=0}^{\infty}\sum_{j=0}^{m}\binom{m}{j}\phi_j\,{}_{L}\mathcal{A}_{m-j}^{[2]}\frac{t^m}{m!} \\ &+\sum_{m=0}^{\infty}\sum_{j=0}^{m}\binom{m}{j}\rho_j\,{}_{L}\mathcal{A}_{m-j}^{[2]}\frac{t^m}{m!}+y\sum_{m=0}^{\infty}{}_{L}\mathcal{A}_{m}^{[2]}\frac{t^m}{m!} \\ &+\left(\sum_{m=0}^{\infty}\frac{(-1)^{m+1}(m+1)\,x^{m+1}t^m}{[(m+1)!]^2}\right)a_1(t)\,a_2(t)\,e^{yt}. \end{aligned}$$

Taking the derivative of both sides of the last equation with respect to x, we get

$$\begin{aligned}\sum_{m=0}^{\infty} D_x\left\{{}_L\mathcal{A}_{m+1}^{[2]}\right\}\frac{t^m}{m!} &= \sum_{m=0}^{\infty}\sum_{j=0}^{m}\binom{m}{j}\phi_j\, D_x\left\{{}_L\mathcal{A}_{m-j}^{[2]}\right\}\frac{t^m}{m!}\\ &+\sum_{m=0}^{\infty}\sum_{j=0}^{m}\binom{m}{j}\rho_j\, D_x\left\{{}_L\mathcal{A}_{m-j}^{[2]}\right\}\frac{t^m}{m!}\\ &+y\sum_{m=0}^{\infty} D_x\left\{{}_L\mathcal{A}_m^{[2]}(x,y)\right\}\frac{t^m}{m!}-\sum_{m=0}^{\infty}{}_L\mathcal{A}_m^{[2]}\frac{t^m}{m!}.\end{aligned}$$

Applying D_x^{-1} to both sides oh the above equation, we get

$$\begin{aligned}\sum_{m=0}^{\infty}{}_L\mathcal{A}_{m+1}^{[2]}\frac{t^m}{m!} &= \sum_{m=0}^{\infty}\sum_{j=0}^{m}\binom{m}{j}\phi_j\,{}_L\mathcal{A}_{m-j}^{[2]}\frac{t^m}{m!}+\sum_{m=0}^{\infty}\sum_{j=0}^{m}\binom{m}{j}\rho_j\,{}_L\mathcal{A}_{m-j}^{[2]}\frac{t^m}{m!}\\ &+y\sum_{m=0}^{\infty}{}_L\mathcal{A}_m^{[2]}\frac{t^m}{m!}-\sum_{m=0}^{\infty}D_x^{-1}{}_L\mathcal{A}_m^{[2]}\frac{t^m}{m!}.\end{aligned} \tag{28}$$

The proof is completed by equating the coefficients of $\frac{t^m}{m!}$ on both sides of the (28).

Theorem 5 *The Laguerre type twice-iterated Appell polynomials satisfy the lowering operator, integro-partial raising operator and integro-partial differential equation provided by*

$${}_yL_m^- : = \frac{1}{m}D_y, \tag{29}$$

$${}_yL_m^+ : = \sum_{j=0}^{m}\frac{\phi_j}{j!}D_y^j+\sum_{j=0}^{m}\frac{\rho_j}{j!}D_y^j+y-D_x^{-1} \tag{30}$$

and

$$\left[\sum_{j=0}^{m}\frac{\phi_j}{j!}D_y^{j+1}+\sum_{j=0}^{m}\frac{\rho_j}{j!}D_y^{j+1}+yD_y-D_x^{-1}D_y-m\right]\mathcal{A}_m^{[2]}=0 \tag{31}$$

where D_x^{-1} is the inverse of D_x, respectively.

Proof Using the following equation:

$$\frac{\partial}{\partial y}\left\{{}_L\mathcal{A}_m^{[2]}\right\}=m\,{}_L\mathcal{A}_{m-1}^{[2]},$$

it clearly shows that the lowering operator is given by

$${}_yL_m^- := \frac{1}{m}D_y.$$

We can write the term $\mathcal{A}_{m-j}^{[2]}$ by applying the lowering operator to the term $\mathcal{A}_{m}^{[2]}$ j times as follows:

$$\begin{aligned} {}_{L}\mathcal{A}_{m-j}^{[2]} &= \left[{}_{y}L_{m-j+1}^{-}\,{}_{y}L_{m-j+2}^{-}\cdots\,{}_{y}L_{m}^{-}\right]{}_{L}\mathcal{A}_{m}^{[2]} \\ &= \left[\frac{1}{(m-j+1)}\frac{1}{(m-j+2)}\cdots\frac{1}{m}\right]{}_{L}\mathcal{A}_{m}^{[2]} \\ &= \frac{(m-j)!}{m!}D_{y}^{j}\left\{{}_{L}\mathcal{A}_{m}^{[2]}\right\}. \end{aligned} \tag{32}$$

Upon substituting from (32) in (25), we get

$${}_{L}\mathcal{A}_{m+1}^{[2]} = \left[\sum_{j=0}^{m}\frac{\phi_{j}}{j!}\,D_{y}^{j} + \sum_{j=0}^{m}\frac{\rho_{j}}{j!}\,D_{y}^{j} + y - D_{x}^{-1}\right]{}_{L}\mathcal{A}_{m}^{[2]}.$$

Hence, the raising operator is given by

$${}_{y}L_{m}^{+} := \sum_{j=0}^{m}\frac{\phi_{j}}{j!}\,D_{y}^{j} + \sum_{j=0}^{m}\frac{\rho_{j}}{j!}\,D_{y}^{j} + y - D_{x}^{-1}.$$

Now, using the factorization method to obtain the differential equation of the Laguerre type twice-iterated Appell polynomials

$${}_{y}L_{m+1}^{-}\,{}_{y}L_{m}^{+}\left({}_{L}\mathcal{A}_{m}^{[2]}\right) = {}_{L}\mathcal{A}_{m}^{[2]},$$

we get

$${}_{L}\mathcal{A}_{m}^{[2]} = \frac{1}{(m+1)}D_{y}\left[\sum_{j=0}^{m}\frac{\phi_{j}}{j!}D_{y}^{j} + \sum_{j=0}^{m}\frac{\rho_{j}}{j!}D_{y}^{j} + y - D_{x}^{-1}\,y\right]{}_{L}\mathcal{A}_{m}^{[2]}$$

and when the terms are reorganized, we can write

$$\left[\sum_{j=0}^{m}\frac{\phi_{j}}{j!}D_{y}^{j+1} + \sum_{j=0}^{m}\frac{\rho_{j}}{j!}D_{y}^{j+1} + yD_{y} - D_{x}^{-1}D_{y} - m\right]{}_{L}\mathcal{A}_{m}^{[2]} = 0 \quad m \geq 1.$$

Thus, the proof is completed.

3. Some particular cases of the Laguerre type twice-iterated Appell polynomials

In this part, we introduce the Laguerre type twice-iterated Euler polynomials, the Laguerre type twice-iterated Bernoulli polynomials and the Laguerre type twice-iterated Bernoulli-Euler polynomials with some particular selections of the functions $a_1(t)$ and $a_2(t)$. We also examine the characteristic properties corresponding to these new subpolynomials.

3.1. Laguerre type twice-iterated Euler polynomials

The Laguerre type twice-iterated Euler polynomials are defined as follows:

$$\left(\frac{2}{e^t+1}\right)^2 e^{yt} C_0(xt) = \sum_{m=0}^{\infty} {}_L E_m^{[2]} \frac{t^m}{m!} \tag{33}$$

where

$$a_1(t) = a_2(t) = \frac{2}{e^t+1}.$$

Corollary 1 *The Laguerre type twice-iterated Euler polynomials satisfy the following recurrence relation*

$$\begin{aligned} {}_L E_{m+1}^{[2]} &= -\sum_{j=0}^{m}\sum_{l=0}^{j} \binom{m}{j}\binom{j}{l} E_{j-l}\, {}_L E_{m-j}^{[2]} \\ &\quad + y\, {}_L E_m^{[2]} - D_x^{-1}\, {}_L E_m^{[2]} \end{aligned} \tag{34}$$

where the Euler numbers E_l *[7] are as follows:*

$$\frac{2}{e^t+1} = \sum_{l=0}^{\infty} E_l \frac{t^l}{l!}. \tag{35}$$

Corollary 2 *The Laguerre type twice-iterated Euler polynomials satisfy the lowering operator, integro-partial raising operator and integro-partial differential equation as follows:*

$${}_y L_m^- : = \frac{1}{m} D_y, \tag{36}$$

$${}_y L_m^+ : = -\sum_{j=0}^{m}\sum_{l=0}^{j} \binom{j}{l} \frac{E_{j-l}}{j!} D_y^j + y - D_x^{-1} \tag{37}$$

and

$$\left[-\sum_{j=0}^{m}\sum_{l=0}^{j} \binom{j}{l} \frac{E_{j-l}}{j!} D_y^{j+1} + y\, D_y - D_x^{-1} D_y - m\right] {}_L E_m^{[2]} = 0 \tag{38}$$

where E_l *are the Euler numbers in* (35)*, respectively.*

Corollary 3 *The Laguerre type twice-iterated Euler polynomials satisfy the following determinantal representation*

$${}_L E_m^{[2]} = (-1)^m \begin{vmatrix} Q_0 & Q_1 & \cdots & Q_{m-1} & Q_m \\ 1 & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \cdots & \binom{m-1}{1}\frac{1}{2} & \binom{m}{1}\frac{1}{2} \\ 0 & 0 & \cdots & \binom{m-1}{2}\frac{1}{2} & \binom{m}{2}\frac{1}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \binom{m}{m-1}\frac{1}{2} \end{vmatrix}, \tag{39}$$

where $\frac{e^t+1}{2} = \sum_{j=0}^{\infty} \delta_j \frac{t^j}{j!}$ *and* $Q_m = \sum_{j=0}^{m} \binom{m}{j} \delta_j\, {}_L E_{m-j}^{[2]}$.

3D surface plots of Laguerre type twice-iterated Euler polynomials are shown in Figure 1 for $\mathrm{E}_{2}^{[2]}(x,y)$ and $\mathrm{E}_{3}^{[2]}(x,y)$.

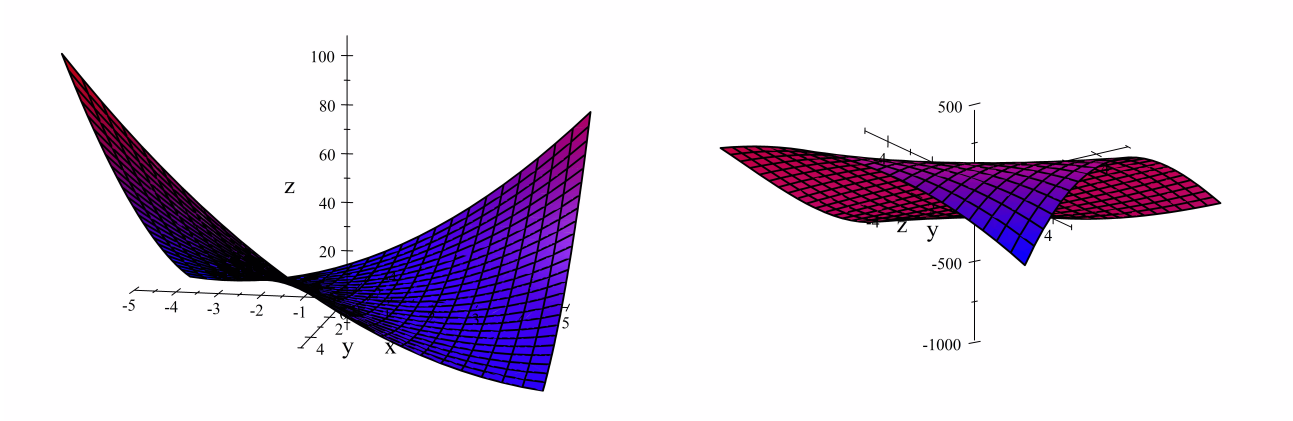


Figure 1. Laguerre type twice-iterated Euler polynomials.

3.2. Laguerre type twice-iterated Bernoulli polynomials

The Laguerre type twice-iterated Bernoulli polynomials are defined as follows:

$$\left(\frac{t}{e^{t}-1}\right)^{2} e^{yt} C_{0}(xt) = \sum_{m=0}^{\infty} {}_{L}B_{m}^{[2]} \frac{t^{m}}{m!} \tag{40}$$

where

$$a_{1}(t) = a_{2}(t) = \frac{t}{e^{t}-1}.$$

We may call the polynomials in (40) as the Laguerre type Bernoulli polynomials of order 2.

Corollary 4 *The Laguerre type twice-iterated Bernoulli polynomials satisfy the following recurrence relation*

$$\begin{aligned} {}_{L}B_{m+1}^{[2]} &= -2\sum_{j=0}^{m}\sum_{l=0}^{j+1}\binom{m}{j}\binom{j+1}{l}\frac{B_{l}}{j+1}\,{}_{L}B_{m-j}^{[2]} \\ &\quad + y\,{}_{L}B_{m}^{[2]} - D_{x}^{-1}\,{}_{L}B_{m}^{[2]} \end{aligned} \tag{41}$$

where the Bernoulli numbers B_{l} [3] are as follows:

$$\frac{t}{e^{t}-1} = \sum_{l=0}^{\infty} B_{l}\frac{t^{l}}{l!}. \tag{42}$$

Corollary 5 *The Laguerre type twice-iterated Bernoulli polynomials satisfy the lowering operator, integro-*

partial raising operator and integro-partial differential equation as follows:

$$ {}_yL_m^- : = \frac{1}{m} D_y, \tag{43} $$

$$ {}_yL_m^+ : = -2 \sum_{j=0}^{m} \sum_{l=0}^{j+1} \binom{j+1}{l} \frac{B_l}{(j+1)!} D_y^j + y - D_x^{-1} \tag{44} $$

and

$$ \left[-2 \sum_{j=0}^{m} \sum_{l=0}^{j+1} \binom{j+1}{l} \frac{B_l}{(j+1)!} D_y^{j+1} + y\, D_y - D_x^{-1} D_y - m \right] {}_L B_m^{[2]} = 0, \tag{45} $$

where B_l are the Bernaulli numbers in (42), respectively.

Remark 1 *It is seen from Corollaries 4 and 5 that the coefficients in the reccurrence relation, lowering and raising operators and integro-differential equation of the Laguerre type Bernoulli polynomials of order 2 is written in terms of Bernoulli numbers (of order 1).*

Corollary 6 *The Laguerre type twice-iterated Bernoulli polynomials satisfy the following determinantal representation*

$$ {}_L B_m^{[2]} = (-1)^m \begin{vmatrix} Q_0 & Q_1 & \cdots & Q_{m-1} & Q_m \\ 1 & \frac{1}{2} & \cdots & \frac{1}{m} & \frac{1}{m+1} \\ 0 & 1 & \cdots & \binom{m-1}{1} \frac{1}{m-1} & \binom{m}{1} \frac{1}{m} \\ 0 & 0 & \cdots & \binom{m-1}{2} \frac{1}{m-2} & \binom{m}{2} \frac{1}{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \binom{m}{m-1} \frac{1}{2} \end{vmatrix}. \tag{46} $$

where $\frac{e^t-1}{t} = \sum_{j=0}^{\infty} \delta_j \frac{t^j}{j!}$ *and* $Q_m = \sum_{j=0}^{m} \binom{m}{j} \delta_j \, {}_L B_{m-j}^{[2]}$.

3D surface plots of Laguerre type twice-iterated Bernoulli polynomials are shown in Figure 2 for $\mathrm{B}_2^{[2]}(x,y)$ and $\mathrm{B}_3^{[2]}(x,y)$.

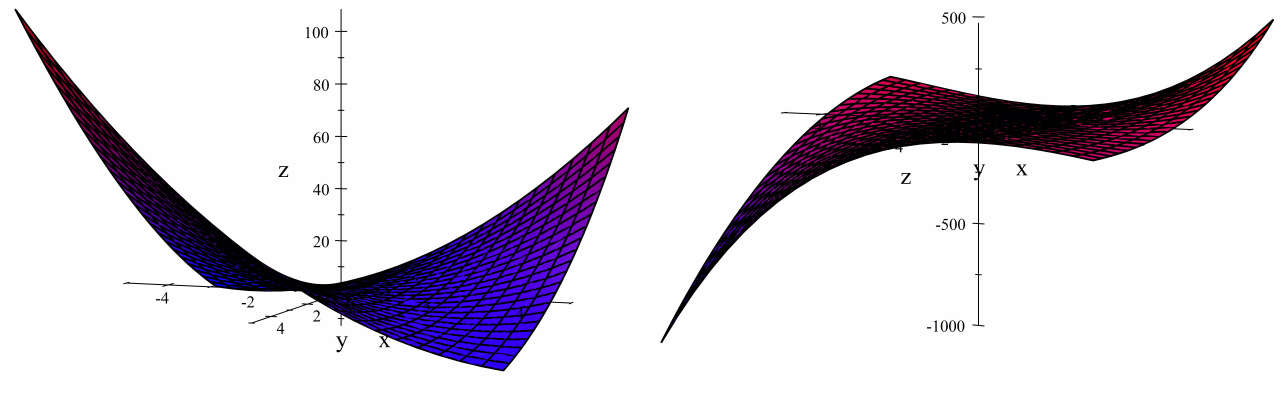


Figure 2. Laguerre type twice-iterated Bernoulli polynomials.

3.3. Laguerre type twice-iterated Bernoulli-Euler polynomials

The Laguerre type twice-iterated Bernoulli-Euler polynomials are defined as follows:

$$\left(\frac{t}{e^t-1}\right)\left(\frac{2}{e^t+1}\right)e^{yt}C_0\left(xt\right)=\sum_{m=0}^{\infty}{}_LBe_m^{[2]}\frac{t^m}{m!} \tag{47}$$

where

$$a_1\left(t\right)=\frac{t}{e^t-1}\qquad\text{and}\qquad a_2\left(t\right)=\frac{2}{e^t+1}.$$

Corollary 7 *The Laguerre type twice-iterated Bernoulli-Euler polynomials satisfy the following recurrence relation*

$$\begin{aligned}{}_LBe_{m+1}^{[2]} &= -\sum_{j=0}^{m}\sum_{l=0}^{j+1}\binom{m}{j}\binom{j+1}{l}\frac{B_l}{j+1}{}_LBe_{m-j}^{[2]}\\ &\quad -\frac{1}{2}\sum_{j=0}^{m}\sum_{l=0}^{j}\binom{m}{j}\binom{j}{l}E_{j-l}\,{}_LBe_{m-j}^{[2]}\\ &\quad +y\,{}_LBe_m^{[2]}-D_x^{-1}{}_LBe_m^{[2]}\end{aligned} \tag{48}$$

where B_l and E_l are defined by (42) and (35), respectively.

Corollary 8 *The Laguerre type twice-iterated Bernoulli-Euler polynomials satisfy the lowering operator, integro-*

partial raising operator and integro-partial differential equation as follows:

$$
\begin{aligned}
{}_yL_m^- : &= \frac{1}{m}D_y, && (49)\\
{}_yL_m^+ : &= -\sum_{j=0}^{m}\sum_{l=0}^{j+1}\binom{j+1}{l}\frac{B_l}{(j+1)!}D_y^j && (50)\\
&\quad -\frac{1}{2}\sum_{j=0}^{m}\sum_{l=0}^{j}\binom{j}{l}\frac{E_{j-l}}{j!}D_y^j + y - D_x^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
&\Bigg[-\sum_{j=0}^{m}\sum_{l=0}^{j+1}\binom{j+1}{l}\frac{B_l}{(j+1)!}D_y^{j+1} - \frac{1}{2}\sum_{j=0}^{m}\sum_{l=0}^{j}\binom{j}{l}\frac{E_{j-l}}{j!}D_y^{j+1}\\
&+ y\,D_y - D_x^{-1}D_y - m\Bigg]\,{}_LBe_m^{[2]} = 0 \qquad (51)
\end{aligned}
$$

where B_l and E_l are defined by (42) and (35), respectively.

Corollary 9 *The Laguerre type twice-iterated Bernoulli-Euler polynomials satisfy the following determinantal representation*

$$
{}_LE_m^{[2]} = (-1)^m \begin{vmatrix}
Q_0 & Q_1 & \cdots & Q_{m-1} & Q_m \\
1 & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\
0 & 1 & \cdots & \binom{m-1}{1}\frac{1}{2} & \binom{m}{1}\frac{1}{2} \\
0 & 0 & \cdots & \binom{m-1}{2}\frac{1}{2} & \binom{m}{2}\frac{1}{2} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & \binom{m}{m-1}\frac{1}{2}
\end{vmatrix} \qquad (52)
$$

where $\frac{e^t+1}{2} = \sum_{j=0}^{\infty}\delta_j\frac{t^j}{j!}$ *and* $Q_m = \sum_{j=0}^{m}\binom{m}{j}\delta_j\,{}_LBe_{m-j}^{[2]}$.

3D surface plots of Laguerre type twice-iterated Bernoulli-Euler polynomials are shown in Figure 3 for $\mathrm{Be}_2^{[2]}(x,y)$ and $\mathrm{Be}_3^{[2]}(x,y)$.

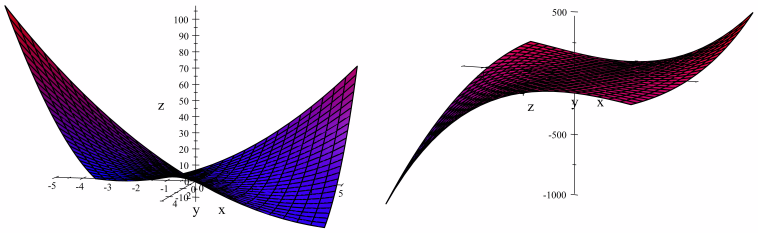


Figure 3. Laguerre type twice-iterated Bernoulli-Euler polynomials.

4. Conclusion

The sequence of Laguerre type twice-iterated Appell polynomials is introduced in this study. Our equivalence theorem for these new polynomials is established. We find some characteristics of the form, including explicit form, recurrence relation, determinantal form, lowering operator, integro-partial raising operator and integral-partial differential equation. For special examples of these new families of polynomials, we present Laguerre type twice-iterated Euler polynomials, Laguerre type twice-iterated Bernoulli polynomials and Laguerre type twice-iterated Bernoulli-Euler polynomials. We also examine the characteristic properties corresponding to these polynomials.

In addition, the Laguerre type r-iterated Appell polynomials $\mathcal{A}_k^{[r]}$ $(r \in \mathbb{N})$ can be defined by the generating function given by

$$a_r(t)\, a_{r-1}(t) \dots a_3(t)\, a_2(t)\, a_1(t)\, e^{yt} C_0(xt) = \sum_{k=0}^{\infty} \mathcal{A}_k^{[r]} \frac{t^k}{k!}. \tag{53}$$

Some properties can be obtained for Laguerre type r-iterated Appell polynomials.

Acknowledgment

We would like to thank The Scientific and Technological Research Council of Türkiye (TÜBİTAK) for the TÜBİTAK BİDEB 2211-A General Domestic Doctorate Scholarship Program that supported the first author. We would like to thank the editor and reviewers for their valuable suggestions and comments.

References

[1] Andrews, LC. Special Functions for Applied Mathematicans and Engineer. MacMillan NewYork, 1985.

[2] Appell, P. Sur une classe de pólynomes. Annales Scientifiques de l'Ecole Normale Supérieure 1880; 9: 119–144 (in French).

[3] Carlitz L. Bernoulli numbers. Fibonacci Quarterly 1968; 6 (3): 71-85.

[4] Dattoli G, Lorenzutta S, Mancho AM, Torre A. Generalized polynomials and associated operational identities. Journal of Computational and Applied Mathematics 1999; 108 (1): 209-218. https://doi.org/10.1016/S0377-0427(99)00111-9

[5] Dattoli G, Torre A. Operatorial methods and two variable Laguerre polynomials. Atti della Accademia delle Scienze di Torino. Classe di Scienze Fisiche, Matematiche e Naturali 1998; 132: 1-7.

[6] Dattoli G, Torre A, Mancho AM. The generalized Laguerre polynomials, the associated Bessel functions and application to propagation problems. Radiation Physics and Chemistry 2000; 59 (3): 229-237. https://doi.org/10.1016/S0969-806X(00)00273-5

[7] Djordjevic GB, Milovanovic GV. Special Classes of Polynomials. Leskovac: University of Nis, Faculty of Technology, 2014.

[8] He MX, Ricci PE. Differential equation of Appell polynomials via the factorization method. Journal of Computational and Applied Mathematics 2002; 139 (2): 231-237. https://doi.org/10.1016/S0377-0427(01)00423-X

[9] Khan S, Al-Saad MW, Khan R. Laguerre-based polynomials: properties and applications. Mathematical and Computer Modelling 2010; 52 (1-2): 247-259. https://doi.org/10.1016/j.mcm.2010.02.022

[10] Khan S, Raza N. General-Appell polynomials within the context of monomiality principle. International Journal of Analysis 2013; 1-11. https://doi.org/10.1155/2013/328032

[11] Khan S, Raza N. 2-iterated Appell polynomials and related numbers. Applied Mathematics Computation 2013; 219 (17): 9469–9483. https://doi.org/10.1016/j.amc.2013.03.082

[12] Khan, S., Riyasat, M., Wani, S. A. On some classes of differential equations and associated integral equations for the Laguerre–Appell polynomials. Advances in Pure and Applied Mathematics 2018; 9 (3): 185-194.

[13] Khan S, Yasmin G, Ahmad N. A note on truncated exponential-based Appell polynomials. Bulletin of the Malaysian Mathematical Sciences Society 2017; 40 (1): 373-388. https://doi.org/10.1007/s40840-016-0343-1

[14] Khan S, Yasmin G, Khan R, Hassan NAM. Hermite-based Appell polynomials: properties and applications. Journal of Mathematical Analysis and Application 2009; 351 (2): 756-764. https://doi.org/10.1016/j.jmaa.2008.11.002

[15] Khan S, Wani SA. Extended Laguerre–Appell polynomials via fractional operators and their determinant forms. Turkish Journal of Mathematics 2018; 42 (4): 1686-1697. https://doi.org/10.3906/mat-1710-55

[16] Özarslan MA, Yılmaz B. A set of finite order differential equations for the Appell polynomials. Journal Computational Applied Mathematics 2014; 259: 108-116. https://doi.org/10.1016/j.cam.2013.08.006

[17] Pintér Á, Srivastava HM. Addition theorems for the Appell polynomials and the associated classes of polynomial expansions. Aequationes Mathematicae 2013; 85 (3): 483-495. https://doi.org/10.1007/s00010-012-0148-8

[18] Ricci PE. Laguerre-type exponentials, Laguerre derivatives and applications. A survey Mathematics 2020; 8 (11): 1-17. https://doi.org/10.3390/math8112054

[19] Ricci PE, Srivastava R. A note on the Laguerre-type Appell and hypergeometric polynomials. Mathematics 2022; 10 (11): 1-11. https://doi.org/10.3390/math10111951

[20] Roman S. The Umbral Calculus. Pure and Applied Mathematics. New York, NY, USA: Academic Press, Inc., 1984.

[21] Sheffer IM. Note on Appell polynomials. Bulletin of the American Mathematical Society 1945; 51 (10): 739-744.

[22] Simsek, Y. Formulas for Poisson–Charlier, Hermite, Milne-Thomson and other type polynomials by their generating functions and p-adic integral approach. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A, Matemáticas 2019; 113: 931-948. https://doi.org/10.1007/s13398-018-0528-6

[23] Srivastava HM, Özarslan MA, Yılmaz B. Some families of differential equations associated with the Hermite-based Appell polynomials and other classes of Hermite-based polynomials. Filomat 2014; 28 (4): 695–708. https://doi.org/10.2298/FIL1404695S

[24] Srivastava HM, Özaraslan MA, Yaşar BY. Difference equations for a class of twice-iterated Δ_h-Appell sequences of polynomials. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A, Matemáticas 2019; 113: 1851-1871. https://doi.org/10.1007/s13398-018-0582-0

[25] Yılmaz B, Özarslan MA. Differential equations for the extended 2D Bernoulli and Euler polynomials. Advances in Difference Equations 2013; 107: 1-16. https://doi.org/10.1186/1687-1847-2013-107