Asymptotic Ergodic Capacity of Multidimensional Vector-Sensor Array MIMO Channels

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***Abstract*—We analyze asymptotic ergodic capacity of multidimensional vector-sensor array MIMO (PMD-MIMO) channels established by the use of dual-polarized antennas in the form of 1D, 2D and/or 3D MIMO arrays. Based on the identification of the decomposition of PMD-MIMO channels into multiple independently-fading and scaled classical MIMO channels in parallel, we consequently derive corresponding asymptotic ergodic capacities analytically via tools out of free probability theory. The analysis of derived asymptotic ergodic capacity expressions in terms of antenna locus aspect ratio β, average symbol SNR per antenna $\bar{\gamma}_s$ and cross-polar discrimination XPD as well as comparison with asymptotic ergodic capacity of classical MIMO channels present important gains in using compact multidimensional vector-sensor array MIMO systems in asymptotic regimes.**

***Index Terms*—MIMO channels, diversity techniques, ergodic capacity, asymptotic analysis.**

I. Introduction

DUE TO THE provision of spatial diversity and multiplexing gain yielding significantly higher achievable data rates and error rate quality especially over fading wireless links, multiple-input multiple-output (MIMO) communication techniques with transmit/receive antenna-arrays is receiving a great attention both in research as well as in commercial deployments for communication systems today. The performance gains of MIMO systems was first presented by Foschini and Gans in [1] and the 1999 paper of Telatar [2] followed by other subsequent contributions in quite large volume on various aspects of MIMO transmission/reception.

The analysis of MIMO systems up-to-date has been generally considered for the construction of linear antenna arrays at the transmitter and receiver respectively, as we will denote throughout the sequel, a classical linear antenna array MIMO channel. Recently only a few number of works attacked the idea to use both space and a combination of polarization and pattern diversity for MIMO channels in a multidimensional setting [3]–[5]. Futhermore, some contributions as [6]–[8] use multimode antennas to exploit multimode-diversity for constructing ultra-compact arrays for MIMO communication between mobile handheld station and the base station.

Author in [9] presented the formulation of dual-polarized antennas based vector-sensor array multidimensional MIMO systems and the derivation of corresponding ergodic capacity for the finite-dimensional case where a wider treatment of the basis formulations for the system model in this paper can also be found. The main difference between the system model proposed in [9] and [3]–[8] is that there exist fading envelope correlation in [3]–[5]. and modal correlation in [6]–[8] while no correlation structure is assumed in [9] taking the interelement distance in all dimensions much larger than $\frac{\lambda}{2}$; i.e. $d_m >> \frac{\lambda}{2}$. Thus, compactness in our construction is devoted with respect to the classical antenna array structure for MIMO communication as presented in the conventional models of [1], [2].

In this sequel, the analysis in [9] is extended to cover the asymptotic scenario where the number of antenna locuses at the transmitter and receiver tends to infinity with a fixed finite aspect ratio β. Such asymptotic approaches are in general used for high-dimensionality or large-system analysis in MIMO and CDMA systems and are also considered to be a good representative for finite-sized system performance expectations [2], [10]–[13].

The sequel is organized as follows. Section 2. presents briefly in referral to [9] the system model of the vector-sensor array multidimensional MIMO channels. Section 3. presents the derivation of asymptotic ergodic capacities followed by numerical results and discussion in Section 4. Section 5. finally presents our conclusions.

II. System Model

In the generalized dual-polarized vector-sensor array MIMO (PMD-MIMO) system with uniform interelement spacing in the form of 1D, 2D and/or 3D MIMO arrays composed of $2t_M \times 2r_M$ single-path Rayleigh-fading links of unity power and with total transmitter power-constraint on the codeword-letters P_M, the discrete-time transfer function matrix in each channel-use has representation over the complex topological tensor-product space that yields the following input-output relation channel-model in each channel-use respectively:

$$\mathbf{Y}_M[n] = \mathbf{H}_M[n] \otimes_2^{\mathcal{C}} \mathbf{X}_M[n] + \mathbf{N}_M[n], \forall n \qquad (1)$$

where M is the maximum of the dimension of array structure between transmitter and receiver; i.e. $M = 1, 2, 3$, $\mathbf{X}_M[n] \in \mathcal{C}^{t_1 \times t_2 \times t_3}$ is the array of input codeword-letters, $\mathbf{Y}_M[n] \in \mathcal{C}^{r_1 \times r_2 \times r_3}$ is the array of soft-output sufficient-statistics for joint maximum-likelihood decoding over receive antennas, and $\mathbf{N}_M[n] \in \mathcal{C}^{r_1 \times r_2 \times r_3}$ is the array of spatially-independent elementwise zero-mean unity-variance circularly-symmetric complex background-AWGN samples with variances of σ_M^2 over all receive antennas respectively. Each interelement link between dual-polarized antenna pairs at the receiver and transmitter in this channel description is topologically described by a 2×2 matrix $\mathbf{H}_M^{(i/j)}[n] = \begin{pmatrix} h_{0,0} & h_{0,1} \\ h_{1,0} & h_{1,1} \end{pmatrix}$ where all

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4 elements are zero-mean and uncorrelated, $\mathcal{E}\left\{|h_{0,0}|^2\right\} = \mathcal{E}\left\{|h_{1,1}|^2\right\} = 1$, $\mathcal{E}\left\{|h_{0,1}|^2\right\} = \mathcal{E}\left\{|h_{1,0}|^2\right\} = \rho$. The term ρ is the *cross-polar mixture ratio* and depicts the ratio of power in the cross-polar components to the power in the co-polar components.

III. Asymptotic Ergodic Capacities

The ergodic capacity of a classical MIMO channel is given in nats/sec/Hz by [2]:

$$C_{(erg)}^{(MIMO)} = \mathcal{E}_{\mathbf{H}_l\mathbf{H}_l^\dagger}\left\{\ln\left\{\left|\mathbf{K}_{\mathbf{w}_l} + \frac{P_l}{t_l}\mathbf{H}_l\mathbf{H}_l^\dagger\right|\right\}\right\} \quad (2)$$

where $|.|$ in (2) stands for the determinant operator. (2) can be equivalently written in terms of the central Wishart class eigenvalue density function of the crosscorrelation matrix of the channel as [14]:

$$C_{(erg)}^{(MIMO)} = \int_0^\infty \ln(1+\gamma_s\lambda) \sum_{k=0}^{m_l-1} \frac{k!}{(k+n_l-m_l)!} \times \left[L_k^{n_l-m_l}(\lambda)\right]^2 \lambda^{n_l-m_l} e^{-\lambda} d\lambda \quad (3)$$

where receiver-side complete CSI and no transmitter-CSI is assumed; hence total-transmitter power P_l is uniformly-distributed over all transmit antennas. Furthermore, $n_l = \max(t_l, r_l)$, $m_l = \min(t_l, r_l)$, $\gamma_s = \frac{1}{t_l}\frac{P_l}{\sigma_l^2} = \frac{1}{t_l}\frac{E_s^{(l)}}{N_0^{(l)}}$, is the symbol SNR per SISO subchannel and $L_k^\alpha(x)$ is the αth-order kth Laguerre polynomial.

Asymptotic ergodic capacity derivation of a classical MIMO channel where n_l and m_l tends to infinity with their ratio tending to the finite aspect, $\beta = \frac{m_l}{n_l}$, further depends on the use of the asymptotic eigenvalue density function for the Wishart ensembles given by the Marčenko-Pastur Law ((1.12) of [15]):

$$f_\beta^{(MP)}(x) = (1-\beta)^+\delta(x) + \frac{\sqrt{(x-a)(b-x)}}{2\pi x} \quad (4)$$

where $a = (1-\sqrt{(}\beta))^2$, $b = (1+\sqrt{(}\beta))^2$, and the asymptotic ergodic capacity for a classical MIMO channel is thus given by:

$$C_{(erg,asy)}^{(MIMO)}(\beta) = \int_a^b \ln(1+\gamma_s\lambda) f_\beta^{(MP)}(\lambda) d\lambda \quad (5)$$

As presented for PMD-MIMO(M) channels in (1), the tensorial representation of the channel model is also previously used in literature for the analysis and derivation of MUSIC algorithms with vector-sensor arrays for polarized seismic sources localization in a facilitative manner [16]. For the derivation of the ergodic capacity for PMD-MIMO(M) channels under receiver-side complete CSI only and no transmitter-CSI, author exploited this facilitative approach in [9] using the underlying hypercomplex manifold structure of the channel model in (1) based on the complex, quaternionic and octonionic bases for the PMD-MIMO(M) channel at $M = 1, 2, 3$ respectively and identified the following property:

Proposition 1.: A single-path Rayleigh-fading ($t_1 \times t_2 \times t_3 = t_M, r_1 \times r_2 \times r_3 = r_M$) PMD-MIMO($M$) channel with noise variance per transmit/receive dual-polarized antennas σ_M^2 and cross-polar mixture ratio ρ is equivalent to 2^{M-1} $(2t_M, 2r_M)$ independently-fading classical MIMO channels in parallel each composed of $m_M = \min(2t_M, 2r_M)$ parallel SISO subchannels scaled with $k(\rho) = \left(\frac{1-\sqrt{\rho}}{1-\rho}\right)^2$ and with equivalent noise variance per subchannel per parallel MIMO channel σ_M^2:

$$\tilde{\mathbf{W}}(M) = k(\rho) \sum_{i=0}^{2^{M-1}-1} \mathbf{H}_i\mathbf{H}_i^H = k(\rho)\mathbf{W}(M) \quad (6)$$

where all individual crosscorrelation matrices in the summation definition of (6) are in central Wishart Ensembles class. Within this observation, the construction of the channel crosscorrelation matrix from a single Wishart matrix in the case of PMD-MIMO(1) channels due to complex bases also validates the applicability of the model in conjunction with the conventional models presented for asymptotic analysis in [2], [10], [11].

To compute the asymptotic ergodic capacity via this decomposition result for PMD-MIMO(M) channels, the asymptotic eigenvalue density of the matrix $\mathbf{W}(M)$ in (6) that is asymptotically a sum of random matrices having Marčenko-Pastur eigenvalue spectrum is required and we have to use the tools out of free probability theory [15]. Free probability theory provides a set of tools in dealing with noncommutative random variables that are asymptotically free such as certain classes of asymptotic random matrices where freeness is the counterpart of the independence notion in classical probability theory.

The central Wishart ensemble class crosscorrelation matrices in the definition of the matrix $\mathbf{W}(M)$ are mutually asymptotically free [15] and the eigenvalue density of a sum of asymptotically free random variables that corresponds to the *additive free convolution* of their corresponding probability measures is computed via R-transform defined as:

$$\mathbf{R}_{\mathbf{A}+\mathbf{B}} = \mathbf{R}_{\mathbf{A}} + \mathbf{R}_{\mathbf{B}} \quad (7)$$

where R-transform is defined as:

$$\mathbf{R}_X(z) = \mathbf{S}_X^{-1}(-z) - \frac{1}{z} \quad (8)$$

in terms of the functional inverse of the Stieltjes transform with respect to functional decomposition; i.e. $z = S_X^{-1}(S_X(z))$. The Stieltjes transform is given by:

$$\mathbf{S}_X(z) = \mathcal{E}\left[\frac{1}{X-z}\right] = \int_{-\infty}^{\infty} \frac{1}{\lambda - z} f_X(\lambda) d\lambda \quad (9)$$

with the inverse transform definition of:

$$f_X(\lambda) = \lim_{w \to 0^+} \frac{1}{\pi} Im\left[\mathbf{S}_X(\lambda + jw)\right] \quad (10)$$

The R-transform of the Marčenko-Pastur Law in (4) is given by (2.80) of [15] as:

$$\mathbf{R}^{(MP)}(z) = \frac{\beta}{1-z} \quad (11)$$

Thus:

$$\mathbf{R}_{\mathbf{W}(M)} = \frac{2^{M-1}\beta}{1-z} \quad (12)$$

with the corresponding Stieltjes transform obtained via (9):

$$\mathbf{S}_{\mathbf{W}(M)}(z) = \frac{-1 + 2^{M-1} - z}{2z} \pm \frac{\sqrt{z^2 - 2(1+2^{M-1})z + (2^{M-1}\beta - 1)^2}}{2z} \quad (13)$$

The asymptotic eigenvalue density of the matrix $\mathbf{W}(M)$ is found via using inverse Stieltjes transform in (10) on (13) as:

$$f_{\mathbf{W}(M)}(x) = (1 - 2^{M-1}\beta)^{+}\delta(x) + \frac{\sqrt{(x-a)(b-x)}}{2\pi x} \quad (14)$$

with $a = (1 - \sqrt{2^{M-1}\beta})^2$ and $b = (1 + \sqrt{2^{M-1}\beta})^2$. The asymptotic ergodic capacity of PMD-MIMO(M) channels is then given in nats/sec/Hz by:

$$C_{(erg,asy)}^{(\mathrm{P}MD-MIMO)}(\beta, M) = \int_a^b \ln\left(1 + k(\rho)\gamma_s\lambda\right) f_{\mathbf{W}(M)}(\lambda)d\lambda \quad (15)$$

with $a = (1-\sqrt{2^{M-1}\beta})^2$ and $b = (1+\sqrt{2^{M-1}\beta})^2$. The integrals in (5) and (15) for classical MIMO and PMD-MIMO(M) respectively yield the asymptotic ergodic capacities via (1.14) of [15] as:

$$C_{(erg,asy)}^{(MIMO)}(\beta, \rho, \gamma_s) = \beta \ln(1 + \gamma_s - \frac{1}{4}\mathcal{F}(\gamma_s, \beta)) + \ln(1 + \gamma_s\beta - \frac{1}{4}\mathcal{F}(\gamma_s, \beta)) - \frac{1}{4\gamma_s}\mathcal{F}(\gamma_s, \beta) \quad (16)$$

$$C_{(erg,asy)}^{(\mathrm{P}MD-MIMO)}(\beta, \rho, \gamma_s, M) = 2^{M-1}\beta \ln(1 + k(\rho)\gamma_s - \frac{1}{4}\mathcal{F}(k(\rho)\gamma_s, 2^{M-1}\beta)) + \ln(1 + 2^{M-1}k(\rho)\gamma_s\beta - \frac{1}{4}\mathcal{F}(k(\rho)\gamma_s, 2^{M-1}\beta)) - \frac{1}{4k(\rho)\gamma_s}\mathcal{F}(k(\rho)\gamma_s, 2^{M-1}\beta) \quad (17)$$

where:

$$\mathcal{F}(x, y) = \left(\sqrt{x(1+\sqrt{y})^2 + 1} - \sqrt{x(1-\sqrt{y})^2 + 1}\right)^2 \quad (18)$$

IV. Numerical Results and Discussion

We present herein the comparative analysis of the asymptotic ergodic capacity of a classical MIMO channel via (16) and the asymptotic ergodic capacity of PMD-MIMO(M) channels in (17) for different values of the key system parameters β, crosspolar discrimination XPD defined as the inverse crosspolar mixture ratio ρ and average symbol SNR per antenna $\bar{\gamma}_s$ that is equal to symbol SNR per antenna γ_s due to unity path power Rayleigh fading links definition between each transmit and receive antenna.

Asymptotic ergodic capacities of classical MIMO and PMD-MIMO(M) channels for $M = 1, 2, 3$ versus aspect ratio β for different crosspolar discrimination XPD values are presented in Fig. 1. Unlike the finite-dimensional scenario, the capacity gain of PMD-MIMO(1) transmission/reception over classical MIMO transmission due to the increased spatial diversity obtained is degraded by crosspolar mixture in the asymptotic regime and the asymptotic capacity of PMD-MIMO(1) presents a lower bound for the capacity of the classical MIMO channel. The capacity gains of PMD-MIMO(2) and PMD-MIMO(3) versus classical MIMO channel at the same aspect ratio are further determined by crosspolar discrimination. When crosspolar discrimination is good such as in Fig. 1a where $XPD = 15dB$, both PMD-MIMO(2) and PMD-MIMO(3) present higher asymptotic ergodic capacity than classical MIMO channel. However, as XPD decreases, crossovers may occur such that above some β value classical MIMO may achieve a higher asymptotic ergodic capacity as in Fig. 1b where $XPD = 5$ dB and as XPD further decreases as in Fig. 1c where $XPD = -5$ dB, classical MIMO may achieve a higher asymptotic ergodic capacity for all values of β than PMD-MIMO(2) and PMD-MIMO(3). The asymptotic ergodic capacity in all four capacity depictions in Fig. 1 also present a linear increase in aspect ratio for large β while there exist a capacity saturation effect for low aspect ratios with linear to logarithmic transition for scaling with β.

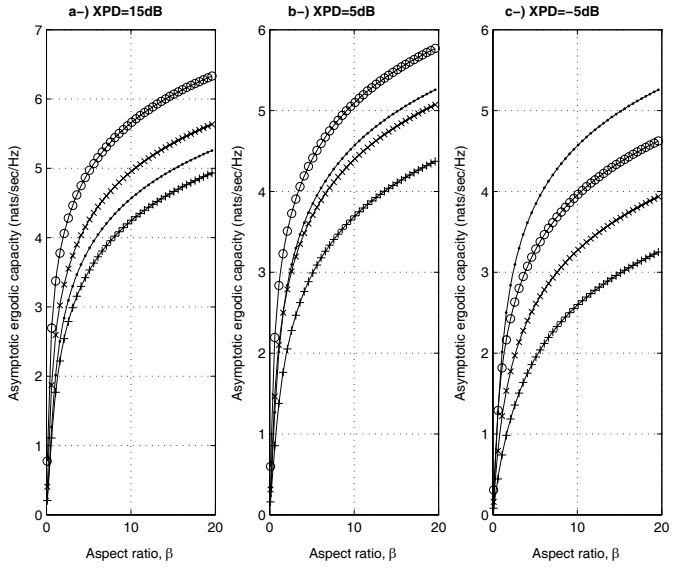


Fig. 1. Asymptotic ergodic capacities versus aspect ratio β at average symbol SNR per antenna $\bar{\gamma}_s = 10$ dB and cross-polar discrimination $XPD = 15, 5, -5$ dB (.=Classical MIMO, +=PMD-MIMO(1), x=PMD-MIMO(2), o=PMD-MIMO(3)).

Asymptotic ergodic capacities versus average symbol SNR per antenna $\bar{\gamma}_s$ in dB are presented in Fig. 2. for different crosspolar discrimination XPD values. A similar comparative analysis of asymptotic ergodic capacity between classical MIMO channel and the PMD-MIMO(M) channels is valid for average SNR analysis as presented for versus β in Fig. 1. There also exists a capacity scaling transition in SNR that is for large SNRs, the capacity scales linearly with SNR while the capacity scaling is higher-order in SNR for low SNRs. Furthermore, in light of the analysis in Fig. 1 and Fig. 2, as the cross-polar discrimination decreases, the capacity gains of PMD-MIMO channels are reduced however, the capacity reduction for a fixed cross-polar mixture is further inversely proportional to the aspect ratio β and average symbol SNR per antenna $\bar{\gamma}_s$.

V. Conclusion

We analyzed in this sequel the asymptotic ergodic capacity of multidimensional vector-sensor array MIMO channels composed of dual-polarized antennas and presented a comparative analysis of the obtained results with the asymptotic ergodic capacity of classical linear antenna array MIMO channels. The analysis of the derived results in terms of the key system

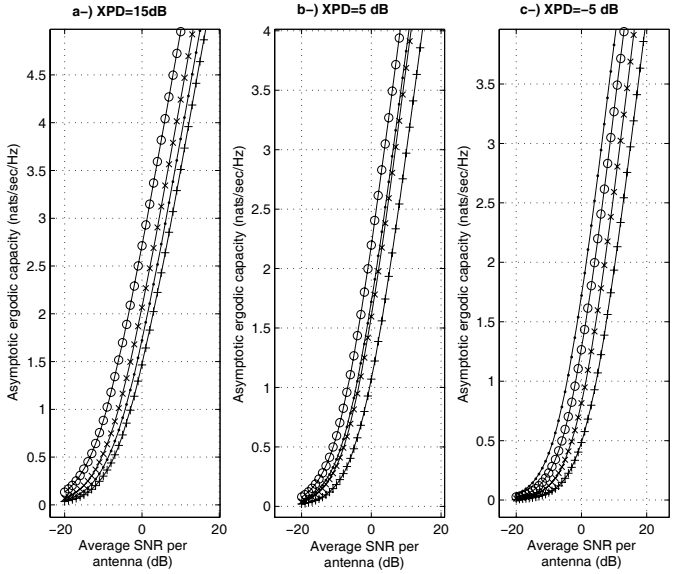


Fig. 2. Asymptotic ergodic capacities versus average symbol SNR per antenna $\bar{\gamma}_s$ at $\beta = 5$ and cross-polar discrimination $XPD = 15, 5, -5$ dB (.=Classical MIMO, +=PMD-MIMO(1), x=PMD-MIMO(2), o=PMD-MIMO(3)).

parameters such as aspect ratio β between the transmit and receive antenna locuses, average symbol SNR per antenna $\bar{\gamma}_s$ and cross-polar discrimination XPD both allows the extraction of capacity scaling nature in terms of these parameters and further presents the capacity gains achievable by spatially-compact two and three dimensional vector-sensor array MIMO channels over classical MIMO systems in asymptotic regimes as well when crosspolar discrimination XPD between dual antennas is good.

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