

Dunkl generalization of Szász Beta type operators

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Abstract

The goal in the paper is to advertise Dunkl extension of Szász beta type operators. We initiate approximation features via acknowledged Korovkin and weighted Korovkin theorem and obtain the convergence rate from the point of modulus of continuity, second order modulus of continuity, the Lipschitz class functions, Peetre's K -functional and modulus of weighted continuity by Dunkl generalization of Szász beta type operators.

Key words: Dunkl type generalization; Szász operators; Peetre's K -functional; Lipschitz functions.

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1 Introduction

Newly, several mathematicians have made many studies concerning generalization of Szász operators (for example, see [2, 3, 8, 10, 16, 18, 20]). Moreover, important definitions, facts and features coupled with approximation theory can be found in [1, 4, 6, 12, 13, 19]. For ν , $x \in [0, \infty)$ and $g \in C[0, \infty)$, in [17], Dunkl analogue of Szász operators is given by

$$S_n^*(g; x) = \frac{1}{e_\nu(nx)} \sum_{r=0}^{\infty} \frac{(nx)^r}{\gamma_\nu(r)} g\left(\frac{r + 2\nu\theta_r}{n}\right); \quad n \in \mathbb{N}. \quad (1)$$

Here the $e_\nu(x)$ is defined as

$$e_\nu(x) = \sum_{r=0}^{\infty} \frac{x^r}{\gamma_\nu(r)}, \quad (2)$$

where for $r \in \mathbb{N}_0$ and $\nu > -\frac{1}{2}$ the coefficients γ_ν are given as follows

$$\gamma_\nu(2r) = \frac{2^{2r} r! \Gamma(r + \nu + 1/2)}{\Gamma(\nu + 1/2)} \quad \text{and} \quad \gamma_\nu(2r + 1) = \frac{2^{2r+1} r! \Gamma(r + \nu + 3/2)}{\Gamma(\nu + 1/2)} \quad (3)$$

in [15]. Also the coefficients γ_ν has the recursion relation

$$\frac{\gamma_\nu(r+1)}{\gamma_\nu(r)} = (2\nu\theta_{r+1} + r + 1), \quad r \in \mathbb{N}_0, \quad (4)$$

where for $p \in \mathbb{N}$, θ_r is given as

$$\theta_r = \begin{cases} 0, & \text{if } r = 2p \\ 1, & \text{if } r = 2p + 1 \end{cases}. \quad (5)$$

Also, the authors gave the other Dunkl generalizations of Szász operators in [9, 14]. Now, for $n \geq 1$, we define a Dunkl analogue of Szász beta type operators defined by

$$T_n(g; x) = \frac{(n-1)}{e_\nu(nx)} \sum_{r=1}^{\infty} \binom{n+r-2}{r-1} \frac{(nx)^r}{\gamma_\nu(r)} \int_0^\infty s^{r-1} (1+s)^{-n-r+1} g(s) ds + \frac{g(0)}{e_\nu(nx)}, \quad (6)$$

where $e_\nu(x)$ and γ_ν are defined in (2) and (3), and $\nu, x \geq 0$. Furthermore $g(s)$ is defined on subset of all continuous functions on $[0, \infty)$ for which the integral exists finitely. Here well-known Beta function is denoted as $B(.,.)$ is given

$$\int_0^\infty s^{r-1} (1+s)^{-n-r+1} ds = B(r, n-1). \quad (7)$$

Lemma 1 Using (7), we derive for $m \in \mathbb{N}$

$$\int_0^\infty s^{r-1} (1+s)^{-n-r+1} s^m ds = B(r+m, n-m-1). \quad (8)$$

Lemma 2 For T_n operators in (6), the important properties are hold.

$$T_n(1; x) = 1, \quad (9)$$

$$|T_n(s; x) - x| \leq \frac{2}{n-2}x + \frac{2\nu}{n-2} \text{ for } n > 2, \quad (10)$$

$$|T_n(s^2; x) - x^2| \leq \frac{5n-6}{n^2-5n+6}x^2 + \left[\frac{4\nu n}{n^2-5n+6} + \frac{2n}{n^2-5n+6} \right] x + \frac{4\nu^2+6\nu}{n^2-5n+6} \text{ for } n > 3, \quad (11)$$

$$\begin{aligned} |T_n(s^3; x) - x^3| &\leq \frac{1}{(n-2)(n-3)(n-4)} \{ (9n^2 - 26n + 24)x^3 + 6n^2(\nu+1)x^2 \\ &\quad + (12\nu^2n + 18\nu n + 6n)x + 12\nu^2 + 4\nu + 8\nu^3 \} \text{ for } n > 4, \end{aligned} \quad (12)$$

$$\begin{aligned} |T_n(s^4; x) - x^4| &\leq \frac{1}{n^4-14n^3+71n^2-154n+120} \{ (14n^3 - 71n^2 + 154n - 120)x^4 + (8\nu n^3 + 12n^3)x^3 \\ &\quad + (62\nu n^2 + 24\nu^2n^2 + 36n^2)x^2 + (32\nu^3n + 96\nu^2n + 56\nu n + 24n)x \\ &\quad + 16\nu^4 + 48\nu^3 + 44\nu^2 + 12\nu \} \text{ for } n > 5. \end{aligned} \quad (13)$$

Lemma 3 For T_n operators, we have

$$\begin{aligned}
\Psi_1 & : = T_n(s-x; x) \leq \left(\frac{2}{n-2}\right)x + \frac{2\nu}{n-2} \text{ for } n > 2, \\
\Psi_2 & : = T_n((s-x)^2; x) \leq \left(\frac{9}{n-3}\right)x^2 + \left(\frac{8\nu n - 12\nu + 2n}{n^2 - 5n + 6}\right)x \text{ for } n > 3, \\
\Psi_3 & : = T_n((s-x)^4; x) \leq \frac{88n^2 - 405n + 450}{(n-2)(n-3)(n-5)}x^4 + \frac{(240n - 480\nu + 856n\nu - 432n^2\nu + 64n^3\nu - 228n^2 + 48n^3)}{n^4 - 14n^3 + 71n^2 - 154n + 120}x^3 \\
& + \frac{(96n^2\nu^2 + 170n^2\nu + 60n^2 - 456n\nu^2 - 684n\nu - 120n + 480\nu^2 + 720\nu)}{n^4 - 14n^3 + 71n^2 - 154n + 120}x^2 \\
& + \frac{(24n - 80\nu + 72n\nu - 240\nu^2 - 160\nu^3 + 144n\nu^2 + 64n\nu^3)}{n^4 - 14n^3 + 71n^2 - 154n + 120}x \\
& + \frac{16\nu^4 + 48\nu^3 + 44\nu^2 + 12\nu}{n^4 - 14n^3 + 71n^2 - 154n + 120} \text{ for } n > 6.
\end{aligned} \tag{14}$$

Theorem 4 For the operators in (6) and any $g \in C[0, \infty) \cap E$, one obtain

$$T_n(g; x) \xrightarrow{\text{uniformly}} g(x)$$

on $A \subset [0, \infty)$ which is each compact set as $n \rightarrow \infty$. Here

$$E := \{g : x \in [0, \infty), \frac{g(x)}{1+x^2} \text{ is convergent as } x \rightarrow \infty\}.$$

Now, we evoke functions in the weighted spaces given on $[0, \infty)$ to touch weighted approximation of our operators:

$$\begin{aligned}
B_\rho(\mathbb{R}^+) & : = \{g : |g(x)| \leq M_g \rho(x)\}, \\
C_\rho(\mathbb{R}^+) & : = \{g : g \in C[0, \infty) \cap B_\rho(\mathbb{R}^+)\}, \\
C_\rho^*(\mathbb{R}^+) & : = \{g : g \in C_\rho(\mathbb{R}^+) \text{ and } \lim_{x \rightarrow \infty} \frac{g(x)}{\rho(x)} = \text{const}\}.
\end{aligned}$$

Here the weight function is called by $\rho(x) = 1 + x^2$ and M_g is a constant based just on the function g . Also we keep in mind the space $C_\rho(\mathbb{R}^+)$ has a norm as $\|g\|_\rho := \sup_{x \geq 0} \frac{|g(x)|}{\rho(x)}$ (see [2]).

Theorem 5 For operators T_n in (6) and each function $g \in C_\rho^*(\mathbb{R}^+)$, one has

$$\lim_{n \rightarrow \infty} \|T_n(g; x) - g(x)\|_\rho = 0.$$

2 Convergence of operators in (6)

Firstly, we remind the Lipschitz class of order α for function g . If $g \in Lip_M(\alpha)$, then g satisfies the inequality

$$|g(s) - g(t)| \leq M |s - t|^\alpha$$

where $s, t \in [0, \infty)$, $0 < \alpha \leq 1$ and $M > 0$.

Theorem 6 *If $h \in Lip_M(\alpha)$, the following inequality*

$$|T_n(h; x) - h(x)| \leq M(\tau_n(x))^{\alpha/2},$$

is hold where $\tau_n(x) = \Psi_2$.

Now, we deal with the space symbolized by $\tilde{C}[0, \infty)$ has uniformly continuous functions on $[0, \infty)$ and modulus of continuity $g \in \tilde{C}[0, \infty)$ is denoted as

$$\omega(g; \delta) := \sup_{\substack{s, t \in [0, \infty) \\ |s-t| \leq \delta}} |g(s) - g(t)|. \quad (16)$$

Theorem 7 *The operators in (6) satisfy the inequality*

$$|T_n(g; x) - g(x)| \leq \left(1 + \sqrt{9x^2 + \left(\frac{8\nu n - 12\nu + 2n}{n-2}\right)x}\right) \omega\left(g; \frac{1}{\sqrt{n}}\right),$$

where $g \in \tilde{C}[0, \infty) \cap E$, ω is modulus of continuity.

Now, we note that the space $C_B[0, \infty)$ is all continuous and bounded functions on $[0, \infty)$. Also

$$C_B^2[0, \infty) = \{g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty)\} \quad (17)$$

and the norm on $C_B^2[0, \infty)$ is defined as

$$\|g\|_{C_B^2[0, \infty)} = \|g\|_{C_B[0, \infty)} + \|g'\|_{C_B[0, \infty)} + \|g''\|_{C_B[0, \infty)}$$

for $\forall g \in C_B^2[0, \infty)$.

Lemma 8 *For $h \in C_B^2[0, \infty)$, one has the inequality*

$$|T_n(h; x) - h(x)| \leq \chi_n(x) \|h\|_{C_B^2[0, \infty)}, \quad (18)$$

where

$$\chi_n(x) = \Psi_1 + \Psi_2. \quad (19)$$

Note that the second order of modulus continuity of g on $C_B[0, \infty)$ is as

$$\omega_2(g; \delta) := \sup_{0 < s \leq \delta} \|g(\cdot + 2s) - 2g(\cdot + s) + g(\cdot)\|_{C_B[0, \infty)}.$$

Thus, we can obtain the following important theorem.

Theorem 9 *The operators in (6) satisfy the following inequality*

$$|T_n(g; x) - g(x)| \leq 2M \left\{ \min \left(1, \frac{\chi_n(x)}{2} \right) \|g\|_{C_B[0, \infty)} + \omega_2 \left(g; \sqrt{\frac{\chi_n(x)}{2}} \right) \right\} \quad (20)$$

where $\forall g \in C_B[0, \infty)$, $x \in [0, \infty)$, M is a positive constant which is not based on n and $\chi_n(x)$ is in (19).

Now, we focus the order of the functions $g \in C_\rho^*(\mathbb{R}^+)$. Atakut and Ispir [2], Ispir [11] defined the weighted of continuity denoted by

$$\Omega(g; \delta) = \sup_{x \in [0, \infty), |h| \leq \delta} \frac{|g(x+h) - g(x)|}{(1+h^2)(1+x^2)}$$

for $g \in C_\rho^*(\mathbb{R}^+)$. This modulus satisfying $\lim_{\delta \rightarrow 0} \Omega(g; \delta) = 0$ and

$$|g(s) - g(x)| \leq 2 \left(1 + \frac{|s-x|}{\delta} \right) (1+\delta^2)(1+x^2)(1+(s-x)^2)\Omega(g; \delta), \quad (21)$$

where $s, x \in [0, \infty)$.

Theorem 10 *The operators in (6) satisfy the following inequality*

$$\sup_{x \in [0, \infty)} \frac{|T_n(g; x) - g(x)|}{(1+x^2)^3} \leq M_\nu \left(1 + \frac{1}{n} \right) \Omega \left(g; \frac{1}{\sqrt{n}} \right),$$

where $g \in C_\rho^*(\mathbb{R}^+)$ and M_ν is a constant which is not based on x .

3 The proofs of the results

Proof of Lemma 2.

For $g(s) = 1$, using (8) and $e_\nu(x)$, we have

$$\begin{aligned} T_n(1; x) &= \frac{(n-1)}{e_\nu(nx)} \sum_{r=1}^{\infty} \frac{(nx)^r}{\gamma_\nu(r)} \binom{n+r-2}{r-1} \int_0^\infty \frac{s^{r-1}}{(1+s)^{n+r-1}} ds + \frac{1}{e_\nu(nx)} \\ &= \frac{1}{e_\nu(nx)} \sum_{r=1}^{\infty} \frac{(nx)^r}{\gamma_\nu(r)} + \frac{1}{e_\nu(nx)} = 1. \end{aligned}$$

For $n > 2$ and $g(s) = s$, using (8), (4) and $e_\nu(x)$, respectively, one derive

$$\begin{aligned}
T_n(s; x) &= \frac{n-1}{e_\nu(nx)} \sum_{r=1}^{\infty} \frac{(nx)^r}{\gamma_\nu(r)} \int_0^\infty \frac{s^r}{(1+s)^{n+r-1}} \binom{n+r-2}{r-1} ds + \frac{g(0)}{e_\nu(nx)} \\
&= \frac{1}{e_\nu(nx)} \sum_{r=1}^{\infty} \frac{(nx)^r}{\gamma_\nu(r)} \frac{(r+2\nu\theta_r-2\nu\theta_r)}{n-2} \\
&= \frac{1}{e_\nu(nx)} \frac{1}{n-2} \sum_{r=1}^{\infty} \frac{(nx)^r}{\gamma_\nu(r-1)} - \frac{1}{e_\nu(nx)} \frac{2\nu}{n-2} \sum_{r=1}^{\infty} \theta_r \frac{(nx)^r}{\gamma_\nu(r)} \\
&= \frac{1}{n-2} \left(nx - \frac{2\nu}{e_\nu(nx)} \sum_{r=1}^{\infty} \theta_r \frac{(nx)^r}{\gamma_\nu(r)} \right).
\end{aligned}$$

Thus, we derive

$$\begin{aligned}
|T_n(s; x) - x| &\leq \frac{1}{n-2} \left(2x + \frac{2\nu}{e_\nu(nx)} \sum_{r=0}^{\infty} \frac{(nx)^r}{\gamma_\nu(r)} \right) \\
&\leq \left(\frac{2}{n-2} \right) x + \frac{2\nu}{n-2}.
\end{aligned}$$

For $n > 3$ and $g(s) = s^2$, using (8), we get

$$\begin{aligned}
T_n(s^2; x) &= \frac{n-1}{e_\nu(nx)} \sum_{r=1}^{\infty} \frac{(nx)^r}{\gamma_\nu(r)} \int_0^\infty \frac{s^{r+1}}{(1+s)^{n+r-1}} \binom{n+r-2}{r-1} ds + \frac{g(0)}{e_\nu(nx)} \\
&= \frac{1}{n^2-5n+6} \frac{1}{e_\nu(nx)} \sum_{r=1}^{\infty} r(r+1) \frac{(nx)^r}{\gamma_\nu(r)}.
\end{aligned}$$

Using

$$r(r+1) = -(r-1)2\nu\theta_r + (r+2\nu\theta_r)(r-1) + 2r$$

and (4), we have

$$\begin{aligned}
T_n(s^2; x) &= \frac{1}{(n^2-5n+6)} \frac{1}{e_\nu(nx)} \left\{ \sum_{r=1}^{\infty} \frac{(nx)^r}{\gamma_\nu(r)} (r+2\nu\theta_r)(r-1) \right. \\
&\quad \left. - 2\nu \sum_{r=1}^{\infty} \theta_r (r-1) \frac{(nx)^r}{\gamma_\nu(r)} + \sum_{r=1}^{\infty} \frac{(nx)^r}{\gamma_\nu(r)} 2r \right\} \\
&= \frac{1}{(n-2)(n-3)} \frac{1}{e_\nu(nx)} \left\{ \sum_{r=0}^{\infty} (r+2\nu\theta_r-2\nu\theta_r) \frac{(nx)^{r+1}}{\gamma_\nu(r)} \right. \\
&\quad \left. - 2\nu \sum_{r=1}^{\infty} \theta_r (r-1) \frac{(nx)^r}{\gamma_\nu(r)} + \sum_{r=1}^{\infty} \frac{(nx)^r}{\gamma_\nu(r)} 2r \right\} \\
&= \frac{1}{n^2-5n+6} \frac{1}{e_\nu(nx)} \left\{ \sum_{r=1}^{\infty} \frac{(nx)^{r+1}}{\gamma_\nu(r-1)} - 2\nu \sum_{r=1}^{\infty} \theta_r \frac{(nx)^{r+1}}{\gamma_\nu(r)} \right. \\
&\quad \left. - 2\nu \sum_{r=1}^{\infty} \theta_r (r-1) \frac{(nx)^r}{\gamma_\nu(r)} + \sum_{r=1}^{\infty} \frac{(nx)^r}{\gamma_\nu(r)} 2r \right\}.
\end{aligned}$$

Therefore, since $\theta_r \leq 1$, we obtain

$$\begin{aligned}
|T_n(s^2; x) - x^2| &\leq \frac{1}{(n^2-5n+6)e_\nu(nx)} \left\{ \left| \sum_{r=1}^{\infty} \frac{(nx)^{r+1}}{\gamma_\nu(r-1)} - (n-2)(n-3)e_\nu(nx)x^2 \right| \right. \\
&\quad \left. + 2\nu \sum_{r=1}^{\infty} \frac{(nx)^{r+1}}{\gamma_\nu(r)} + 2\nu \sum_{r=1}^{\infty} (r-1) \frac{(nx)^r}{\gamma_\nu(r)} + \sum_{r=1}^{\infty} \frac{(nx)^r}{\gamma_\nu(r)} 2r \right\} \\
&\leq \frac{1}{(n-2)(n-3)} \frac{1}{e_\nu(nx)} \left\{ |n^2x^2e_\nu(nx) - (n-2)(n-3)e_\nu(nx)x^2| + 2\nu nxe_\nu(nx) \right. \\
&\quad \left. + 2\nu \sum_{r=1}^{\infty} \frac{(nx)^r}{\gamma_\nu(r)} (r + 2\nu\theta_r - 2\nu\theta_r - 1) + 2 \sum_{r=1}^{\infty} \frac{(nx)^r}{\gamma_\nu(r)} (r + 2\nu\theta_r - 2\nu\theta_r) \right\} \\
&\leq \frac{1}{n^2-5n+6} \frac{1}{e_\nu(nx)} \left\{ |n^2x^2e_\nu(nx) - (n-2)(n-3)e_\nu(nx)x^2| \right. \\
&\quad \left. + 2\nu nxe_\nu(nx) + 2\nu(nx + (2\nu + 1))e_\nu(nx) + 2(nx + 2\nu)e_\nu(nx) \right\} \\
&= \frac{5n-6}{n^2-5n+6} x^2 + \left[\frac{4\nu n}{n^2-5n+6} + \frac{2n}{n^2-5n+6} \right] x + \frac{4\nu^2+6\nu}{n^2-5n+6}.
\end{aligned}$$

Similarly, (12) and (13) can be proved.

Proof of Theorem 4. As $n \rightarrow \infty$, under favour of Korovkin Theorem in [12], one has $T_n(g; x) \xrightarrow{\text{uniformly}} g(x)$ on $A \subset [0, \infty)$ which is each compact set because $\lim_{n \rightarrow \infty} T_n(e_i; x) = x^i$, for $e_i = s^i$, $i = 0, 1, 2$, which is uniformly on $A \subset [0, \infty)$ with the help of using Lemma 2.

Proof of Theorem 5. From (9), we can write $\lim_{n \rightarrow \infty} \|T_n(1; x) - 1\|_\rho = 0$. For $n > 2$, by (10) and the following calculation

$$\begin{aligned}
\sup_{x \in [0, \infty)} \frac{|T_n(s; x) - x|}{1 + x^2} &\leq \frac{1}{n-2} \left(2 \sup_{x \in [0, \infty)} \frac{x}{1 + x^2} + 2\nu \sup_{x \in [0, \infty)} \frac{1}{1 + x^2} \right) \\
&\leq \left(\frac{2}{n-2} \right) + \frac{2\nu}{n-2},
\end{aligned}$$

we get

$$\lim_{n \rightarrow \infty} \|T_n(s; x) - x\|_\rho = 0.$$

Finally, for $n > 3$, by (11) and the following calculation

$$\begin{aligned}
\sup_{x \in [0, \infty)} \frac{|T_n(s^2; x) - x^2|}{1 + x^2} &\leq \frac{1}{(n^2-5n+6)} \left\{ (5n-6) \sup_{x \in [0, \infty)} \frac{x^2}{1 + x^2} \right. \\
&\quad \left. + [4\nu n + 2n] \sup_{x \in [0, \infty)} \frac{x}{1 + x^2} \right. \\
&\quad \left. + (4\nu^2 + 6\nu) \sup_{x \in [0, \infty)} \frac{1}{1 + x^2} \right\} \\
&= \frac{1}{(n^2-5n+6)} \left\{ (5n-6) + (2\nu n + n) + (4\nu^2 + 6\nu) \right\},
\end{aligned}$$

we have

$$\lim_{n \rightarrow \infty} \|T_n(s^2; x) - x^2\|_\rho = 0.$$

Thus, we get $\lim_{n \rightarrow \infty} \|T_n(g; x) - g(x)\|_\rho = 0$ for each $g \in C_\rho^*(\mathbb{R}^+)$ via weighted Korovkin-type theorem given by Gadzhiev [7].

Proof of Theorem 6. Using $h \in Lip_M(\alpha)$ and linearity, one has

$$\begin{aligned} |T_n(h; x) - h(x)| &\leq T_n(|h(s) - h(x)|; x) \\ &\leq MT_n(|s - x|^\alpha; x). \end{aligned}$$

From Lemma 2 and Hölder's famous inequality, we derive

$$|T_n(h; x) - h(x)| \leq M[\Psi_2]^{\frac{\alpha}{2}}.$$

Then choosing $\tau_n(x) = \Psi_2$, thus one has the required inequality.

Proof of Theorem 7. By the property of modulus of continuity and (14), one get

$$\begin{aligned} |T_n(g; x) - g(x)| &\leq T_n(|g(s) - g(x)|; x) \\ &\leq \left(1 + \frac{1}{\delta} T_n(|s - x|; x)\right) \omega(g; \delta) \\ &\leq \left(1 + \frac{1}{\delta} \sqrt{\Psi_2}\right) \omega(g; \delta). \end{aligned}$$

Then using Cauchy-Schwarz's famous inequality, one has

$$|T_n(g; x) - g(x)| \leq \left(1 + \frac{1}{\delta} \sqrt{\frac{9}{n-3}x^2 + \left(\frac{8\nu n - 12\nu + 2n}{n^2 - 5n + 6}\right)x}\right) \omega(g; \delta). \quad (22)$$

Choosing $\delta = \frac{1}{\sqrt{n}}$, the proof is done.

Proof of Lemma 8. Using the Taylor's series of the function h , we can write

$$h(s) = h(x) + (s - x)h'(x) + \frac{(s - x)^2}{2!}h''(\varrho), \quad \varrho \in (x, s).$$

From the linear operator, we give

$$T_n(h; x) - h(x) = h'(x)\Psi_1 + \frac{h''(\varrho)}{2}\Psi_2.$$

Then for $n > 3$, using Lemma 3, one obtain

$$\begin{aligned} |T_n(h; x) - h(x)| &\leq \frac{2}{n-2}(x + \nu)\|h'\|_{C_B[0, \infty)} \\ &\quad + \left[\left(\frac{9}{n-3}\right)x^2 + \left(\frac{8\nu n - 12\nu + 2n}{n^2 - 5n + 6}\right)x\right]\|h''\|_{C_B[0, \infty)} \\ &\leq [\Psi_1 + \Psi_2]\|h\|_{C_B^2[0, \infty)}. \end{aligned}$$

Choosing $\chi_n(x) = [\Psi_1 + \Psi_2]$ which finishes the proof.

Proof of Theorem 9. For any $f \in C_B^2[0, \infty)$, from the triangle inequality and Lemma 8, one has

$$\begin{aligned} \Lambda &= |T_n(g; x) - g(x)| \leq |T_n(g - f; x)| + |T_n(f; x) - f(x)| + |g(x) - f(x)| \\ &\leq 2 \|g - f\|_{C_B[0, \infty)} + \chi_n(x) \|f\|_{C_B^2[0, \infty)} \\ &= 2 \left\{ \|g - f\|_{C_B[0, \infty)} + \frac{\chi_n}{2}(x) \|f\|_{C_B^2[0, \infty)} \right\}. \end{aligned}$$

With the help of Peetre's K functional in [6], one has

$$\Lambda \leq 2K \left(g; \frac{\chi_n(x)}{2} \right).$$

Thus we can write

$$\Lambda \leq 2M \left\{ \min \left(1, \frac{\chi_n(x)}{2} \right) \|g\|_{C_B[0, \infty)} + \omega_2 \left(g; \sqrt{\frac{\chi_n(x)}{2}} \right) \right\}$$

because of the well-known connection between K_2 and ω_2 in [6]. We note that the connection is as

$$K_2(g; \delta) \leq C \left\{ \min(1, \delta) \|g\|_{C_B[0, \infty)} + \omega_2 \left(g; \sqrt{\delta} \right) \right\}.$$

Here C is an positive constant [5].

Proof of Theorem 10.

Under favour of (21) and the property of linearity of operator, one has

$$\begin{aligned} \Lambda &= |T_n(g; x) - g(x)| \leq T_n(|g(s) - g(x)|; x) \\ &\leq 2(1 + \delta^2)(1 + x^2)\Omega(g; \delta)T_n \left(\left(1 + \frac{|s - x|}{\delta} \right) (1 + (s - x)^2); x \right) \\ &= 2(1 + \delta^2)(1 + x^2)\Omega(g; \delta) \left\{ T_n(1; x) + \frac{1}{\delta}T_n(|s - x|; x) \right. \\ &\quad \left. + T_n((s - x)^2; x) + \frac{1}{\delta}T_n(|s - x|^3; x) \right\}. \end{aligned}$$

Now, if we apply Cauchy-Schwarz's inequality for $T_n(|s - x|; x)$ and $T_n(|s - x|^3; x)$, then we derive

$$\begin{aligned} T_n(|s - x|; x) &\leq \sqrt{T_n((s - x)^2; x)}, \\ T_n(|s - x|^3; x) &\leq \sqrt{T_n((s - x)^2; x)} \sqrt{T_n((s - x)^4; x)}. \end{aligned}$$

Thus, we get

$$\Lambda \leq 2(1 + \delta^2)(1 + x^2)\Omega(g; \delta) \left(1 + \frac{1}{\delta} \sqrt{\Psi_2} + \Psi_2 + \frac{1}{\delta} \sqrt{\Psi_2 \Psi_3} \right).$$

With the help of (14) and (15), one has

$$\begin{aligned} \Lambda \leq & 2(1 + \delta^2)(1 + x^2)\Omega(g; \delta) \left\{ 1 + \frac{9}{n-3}x^2 + \left(\frac{8\nu n - 12\nu + 2n}{n^2 - 5n + 6} \right) x \right. \\ & + \frac{1}{\delta} \sqrt{\frac{9}{n-3}x^2 + \left(\frac{8\nu n - 12\nu + 2n}{n^2 - 5n + 6} \right) x} \\ & \left. + \frac{1}{\delta} \sqrt{\left(\frac{9}{n-3}x^2 + \left(\frac{8\nu n - 12\nu + 2n}{n^2 - 5n + 6} \right) x \right) \Psi_3} \right\}. \end{aligned}$$

Choosing $\delta = \frac{1}{\sqrt{n}}$, then the proof is completed.

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