AIP Conference Proceedings

Fixed point theorems on orthogonal metric spaces via altering distance functions

Cite as: AIP Conference Proceedings **2183**, 040011 (2019); https://doi.org/10.1063/1.5136131
Published Online: 06 December 2019

Nurcan Bilgili Gungor, and Duran Turkoglu

View Online Export Citation

ARTICLES YOU MAY BE INTERESTED IN

On intuitionistic fuzzy cone symmetric spaces
AIP Conference Proceedings **2183**, 030014 (2019); https://doi.org/10.1063/1.5136118

Chain connectedness
AIP Conference Proceedings **2183**, 030015 (2019); https://doi.org/10.1063/1.5136119

Relative selective versions of the ccc property
AIP Conference Proceedings **2183**, 020004 (2019); https://doi.org/10.1063/1.5136101



AIP Publishing

© 2019 Author(s).

Fixed Point Theorems on Orthogonal Metric Spaces via Altering Distance Functions

Nurcan Bilgili Gungor [1,a)] and Duran Turkoglu [2,b)]

[1]*Department of Mathematics, Faculty of Science and Arts, Amasya University, 05000, Amasya, Turkey*
[2]*Department of Mathematics, Faculty of Science, Gazi University, 06500, Ankara, Turkey*

[a)]Corresponding author: bilgilinurcan@gmail.com, nurcan.bilgili@amasya.edu.tr
[b)]dturkoglu@gazi.edu.tr

Abstract. In 1984, Khan et al. [3] established some fixed point theorems in complete and compact metric spaces by using altering distance functions. In 2017, Gordji et al. [2] described the notion of orthogonal set and orthogonal metric spaces. In their work, they held a generalization of Banach fixed point theorem in this interestingly-defined construction and in addition, applied their acquired results to demonstrate the existence of a solution of an ordinary differential equation. In this paper, some fixed point theorems on orthogonal metric spaces via altering distance functions inspired by [3] and [2] are presented .
Keywords: altering distance, fixed point, orthogonal metric space
PACS: 02.30.Tb, 02.40.Pc

Introduction and Preliminaries

Definition 1 *([2]) Let $X \neq \emptyset$ and $\perp \subseteq X \times X$ be a binary relation. If $\perp$ satisfies the following condition*

$$\exists x_0 \in X; (\forall y \in X, y \perp x_0) \vee (\forall y \in X, x_0 \perp y), \tag{1}$$

it is called an orthogonal set (shortly O-set). And $(X, \perp)$ is called O-set. And the element x_0 is called an orthogonal element.

Definition 2 *([2]) Let $(X, \perp)$ be an orthogonal set (O-set). Any two elements $x, y \in X$ are said to be orthogonally related if $x \perp y$.*

Definition 3 *([2]) A sequence $\{x_n\}$ is called orthogonal sequence (shortly O-sequence) if*

$$(\forall n \in \mathbb{N}; x_n \perp x_{n+1}) \vee (\forall n \in \mathbb{N}; x_{n+1} \perp x_n). \tag{2}$$

Similarly, a Cauchy sequence $\{x_n\}$ is said to be an orthogonally Cauchy sequence (shortly O-Cauchy sequence) if

$$(\forall n \in \mathbb{N}; x_n \perp x_{n+1}) \vee (\forall n \in \mathbb{N}; x_{n+1} \perp x_n). \tag{3}$$

Definition 4 *([2]) Let $(X, \perp)$ be an orthogonal set and d be a usual metric on X. Then $(X, \perp, d)$ is called an orthogonal metric space (shortly O-metric space).*

Definition 5 *([2]) An orthogonal metric space $(X, \perp, d)$ is said to be a complete O-metric space (O-complete) if every O-Cauchy sequence converges in X.*

Definition 6 *([2]) Let $(X, \perp, d)$ be an orthogonal metric space. A function $f : X \to X$ is said to be orthogonally continuous ($\perp$-continuous) at x if for each O-sequence $\{x_n\}$ converging to x implies $fx_n \to fx$ as $n \to \infty$. Also f is $\perp$-continuous on X if f is $\perp$-continuous at every $x \in X$.*

Third International Conference of Mathematical Sciences (ICMS 2019)
AIP Conf. Proc. 2183, 040011-1–040011-4; https://doi.org/10.1063/1.5136131
Published by AIP Publishing. 978-0-7354-1930-8/$30.00

Definition 7 *([2]) Let $(X, \perp, d)$ be an orthogonal metric space and $\lambda \in \mathbb{R}$ with $0 < \lambda < 1$ where $\mathbb{R}$ is the set of real numbers. A function $f : X \to X$ is said to be orthogonal contraction ($\perp$-contraction) with Lipschitz constant λ if*

$$d(fx, fy) \leq \lambda d(x, y) \tag{4}$$

for all $x, y \in X$ whenever $x \perp y$.

Definition 8 *([2]) Let $(X, \perp, d)$ be an orthogonal metric space. A function $f : X \to X$ is called orthogonal preserving ($\perp$-preserving) if $fx \perp fy$ whenever $x \perp y$.*

Theorem 9 *Let $(X, \perp, d)$ be an O-complete metric space (not necessarily complete metric space) and $0 < \lambda < 1$. Let $f : X \to X$ be $\perp$-continuous , $\perp$-contraction (with Lipschitz constant λ) and $\perp$-preserving. Then f has a unique fixed point $x^* \in X$ and is a Picard operator, that is, $\lim_{n\to\infty}(x) = x^*$ for all $x \in X$.*

Definition 10 *([3]) A function $\phi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:*
(i) $\phi(t)$ is continuous and nondecreasing,
(ii) $\phi(t) = 0$ if and only if $t = 0$.
We denote Φ as the set of altering distance functions ϕ.

Main Results

Theorem 11 *Let $(X, \perp, d)$ be an O-complete metric space(not necessarily complete metric space),$f : X \to X$ be a self map , $\phi \in \Phi$ be a sub-additive function and $\alpha, \beta, \gamma : \mathbb{R}^+ - \{0\} \to [0, 1)$ be three decreasing functions such that $(\alpha + 2\beta + \gamma)(t) < 1$ for every $t > 0$. Suppose that f is $\perp$-preserving self mapping satisfying the inequality*

$$\begin{aligned}\phi(d(fx, fy)) \quad &\leq \alpha(d(x, y))\phi(d(x, y)) + \beta(d(x, y))[\phi(d(x, fx)) + \phi(d(y, fy))] \\ &\quad +\gamma(d(x, y)) \min\{\phi(d(x, fy), \phi(d(y, fx))\}\end{aligned} \tag{5}$$

for all $x, y \in X$ where $x \perp y$ and $x \neq y$. In this case, there exists a point $x^* \in X$ such that for any orthogonal element $x_0 \in X$, the iteration sequence $\{f^n x_0\}$ converges to this point. Also, if f is $\perp$-continuous at $x^* \in X$, then $x^* \in X$ is a unique fixed point of f.

Proof 1 *Since $(X, \perp)$ is an O-set,*

$$\exists x_0 \in X; (\forall y \in X, y \perp x_0) \vee (\forall y \in X, x_0 \perp y). \tag{6}$$

And since f is a self mapping on X, for any orthogonal element $x_0 \in X$, $x_1 \in X$ can be chosen as $x_1 = fx_0$. Thus,

$$x_0 \perp f_0 \vee fx_0 \perp x_0 \Rightarrow x_0 \perp x_1 \vee x_1 \perp x_0. \tag{7}$$

Then, if we continue in the same way, so that $\{f^n x_0\}$ is an iteration sequence.
If $x_n = x_{n+1}$ for any $n \in \mathbb{N}$, then we get $x_n = fx_n$ and so f has a fixed point. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since f is $\perp$-preserving, $\{f^n x_0\}$ is an O-sequence and by using inequality (5)

$$\begin{aligned}\phi(d(x_{n+1}, x_n)) \quad &= \phi(d(fx_n, fx_{n-1})) \\ &\leq \alpha(d(x_n, x_{n-1}))\phi(d(x_n, x_{n-1})) + \beta(d(x_n, x_{n-1}))[\phi(d(x_n, x_{n+1})) + \phi(d(x_{n-1}, x_n))] \\ &\quad +\gamma(d(x_n, x_{n-1})) \min\{\phi(d(x_n, x_n)), \phi(d(x_{n-1}, x_{n+1}))\}\end{aligned} \tag{8}$$

and so

$$\phi(d(x_{n+1}, x_n)) \leq \frac{\alpha(d(x_n, x_{n-1})) + \beta(d(x_n, x_{n-1}))}{1 - \beta(d(x_n, x_{n-1}))}\phi(d(x_n, x_{n-1})) < \phi(d(x_n, x_{n-1})). \tag{9}$$

Since $\phi \in \Phi$, $\{d(x_{n+1}, x_n)\}$ is a sequence of decreasing nonnegative real numbers. So there is a $r \geq 0$ and $\lim_{n\to\infty} d(x_{n+1}, x_n) = r$. We will show that $r = 0$. Suppose, on the contrary, that $r > 0$. In this case, by taking the limit $n \to \infty$ in inequality (9) and the continuoty of ϕ, we obtain

$$\phi(r) \leq \frac{\alpha(r) + \beta(r)}{1 - \beta(r)}\phi(r) < \phi(r). \tag{10}$$

This is a contradiction. Therefore we get $r = 0$. Next, we will prove that $\{x_n\}$ is an O-Cauchy sequence. If $\{x_n\}$ is not an O-Cauchy sequence, there exists $\varepsilon > 0$ and the corresponding subsequences $\{k(n)\}$ and $\{l(n)\}$ of $\mathbb{N}$ satisfying $k(n) > l(n) > n$ for which

$$d(x_{k(n)}, x_{l(n)}) \geq \varepsilon \tag{11}$$

and where $k(n)$ is chosen as the smallest integer satisfying (11), that is

$$d(x_{k(n)-1}, x_{l(n)}) < \varepsilon. \tag{12}$$

By (11),(12) and triangular inequality of d, we easily derive that

$$\varepsilon \leq d(x_{k(n)}, x_{l(n)}) \leq d(x_{k(n)}, x_{k(n)-1}) + d(x_{k(n)-1}, x_{l(n)}) < d(x_{k(n)}, x_{k(n)-1}) + \varepsilon. \tag{13}$$

Letting $n \to \infty$, by using $\lim_{n\to\infty} d(x_{n+1}, x_n) = r$ we get

$$\lim_{n\to\infty} d(x_{k(n)}, x_{l(n)}) = \varepsilon. \tag{14}$$

Also, for each $n \in \mathbb{N}$, by using the triangular inequality of d,

$$\begin{aligned} d(x_{k(n)}, x_{l(n)}) - d(x_{k(n)}, x_{k(n)+1}) - d(x_{l(n)+1}, x_{l(n)}) &\leq d(x_{k(n)+1}, x_{l(n)+1}) \\ &\leq d(x_{k(n)}, x_{k(n)+1}) + d(x_{k(n)}, x_{l(n)}) + d(x_{l(n)+1}, x_{l(n)}). \end{aligned} \tag{15}$$

Taking the limit as the $n \to \infty$ in the last inequality we obtain

$$d(x_{k(n)+1}, x_{l(n)+1}) = \varepsilon. \tag{16}$$

Using the inequality (5) and the triangular inequality of d,

$$\begin{aligned} \phi(d(x_{k(n)+1}, x_{l(n)+1})) &= \phi(d(fx_{k(n)}, fx_{l(n)})) \\ &\leq \alpha(d(x_{k(n)}, x_{l(n)}))\phi(d(x_{k(n)}, x_{l(n)})) + \beta(d(x_{k(n)}, x_{l(n)}))[\phi(d(x_{k(n)+1}, x_{k(n)})) + \phi(d(x_{l(n)+1}, x_{l(n)}))] \\ &\quad +\gamma(d(x_{k(n)}, x_{l(n)})) \min\{\phi(d(x_{k(n)}, x_{l(n)+1})) + \phi(d(x_{l(n)}, x_{k(n)+1}))\} \\ &\leq \alpha(\varepsilon)\phi(d(x_{k(n)}, x_{l(n)})) + \beta(\varepsilon)[\phi(d(x_{k(n)+1}, x_{k(n)})) + \phi(d(x_{l(n)+1}, x_{l(n)}))] \\ &\quad +\gamma(\varepsilon)\phi(d(x_{k(n)}, x_{l(n)}) + d(x_{k(n)+1}, x_{k(n)}) + d(x_{l(n)+1}, x_{l(n)})). \end{aligned} \tag{17}$$

Taking the limit as the $n \to \infty$ in the last inequality we obtain

$$\phi(\varepsilon) \leq \{\alpha(\varepsilon) + \gamma(\varepsilon)\}\phi(\varepsilon) < \phi(\varepsilon). \tag{18}$$

It is a contradiction. Therefore $\{x_n\}$ is a O-Cauchy sequence. By the O-completeness of X, there exists $x^* \in X$ such that $\{x_n\} = \{f^n x_0\}$ converges to this point.
Now we show that x^* is a fixed point of f when f is $\perp$-continuous at $x^* \in X$. Assume that f is $\perp$-continuous at $x^* \in X$. Thus,

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = fx^*. \tag{19}$$

so $x^* \in X$ is a fixed point of f.
Now we can show the uniqueness of the fixed point. Suppose that there exist two distinct fixed points x^* and y^*.Then,
(i) If $x^* \perp y^* \vee y^* \perp x^*$, by using the inequality (5)

$$\begin{aligned} \phi(d(x^*, y^*)) &= \phi(d(fx^*, fy^*)) \\ &\leq \alpha(d(x^*, y^*))\phi(d(x^*, y^*)) + \beta(d(x^*, y^*))[\phi(d(x^*, Tx^*)) + \phi(d(x^*, y^*))] \\ &\quad +\gamma(d(x^*, y^*)) \min\{\phi(d(x^*, Ty^*)), \phi(d(y^*, Tx^*))\} \\ &\leq \{\alpha(d(x^*, y^*) + \gamma(d(x^*, y^*)\}\phi(d(x^*, y^*)) \\ &< \phi(d(x^*, y^*)) \end{aligned} \tag{20}$$

which is a contradiction so that $x^* \in X$ is a unique fixed point of f.

(ii) If not, for the chosen orthogonal element $x_0 \in X$,

$$[(x_0 \perp x^*) \wedge (x_0 \perp y^*)] \vee [(x^* \perp x_0) \wedge (y^* \perp x_0)] \tag{21}$$

and since f is $\perp$- preserving,

$$[(fx_n \perp x^*) \wedge (fx_n \perp y^*)] \vee [(x^* \perp fx_n) \wedge (y^* \perp fx_n)] \tag{22}$$

is obtained. Now, by using the triangular inequality of d, ϕ is nondecreasing sub-additive function and the inequality (5)

$$\begin{aligned}
\phi(d(x^*, y^*)) &= \phi(d(fx^*, fy^*)) \\
&\leq \phi(d(fx^*, fx_{n+1}) + d(fx_{n+1}, fy^*)) \\
&\leq \phi(d(fx^*, f(fx_n))) + \phi((f(fx_n), fy^*)) \\
&\leq \alpha(d(x^*, fx_n))\phi(d(x^*, fx_n)) + \beta(d(x^*, fx_n))[\phi(d(x^*, fx^*)) + \phi(d(fx_n, fx_{n+1}))] \\
&\quad +\gamma(d(x^*, fx_n)) \min\{\phi(d(x^*, fx_{n+1})), \phi(d(fx_n, fx^*))\} \\
&\quad +\alpha(d(y^*, fx_n))\phi(d(y^*, fx_n)) + \beta(d(y^*, fx_n))[\phi(d(y^*, fy^*)) + \phi(d(fx_n, fx_{n+1}))] \\
&\quad +\gamma(d(y^*, fx_n)) \min\{\phi(d(y^*, fx_{n+1})), \phi(d(fx_n, fy^*))\}
\end{aligned} \tag{23}$$

and taking limit $n \to \infty$, we get that $x^* = y^*$. Thus, $x^* \in X$ is a unique fixed point of f.

If we assume $b = c = 0$ in Theorem 11, we get the following corollary:

Corollary 12 *Let $(X, \perp, d)$ be an O-complete metric space (not necessarily complete metric space) ,$f : X \to X$ be a self map , $\phi \in \Phi$ be a sub-additive function and $\alpha : \mathbb{R}^+ - \{0\} \to [0, 1)$ be a decreasing function such that $\alpha(t) < 1$ for every $t > 0$. Suppose that f is $\perp$-preserving self mapping satisfying the inequality*

$$\phi(d(fx, fy)) \leq \alpha(d(x, y))\phi(d(x, y)) \tag{24}$$

for all $x, y \in X$ where $x \perp y$ and $x \neq y$. In this case, there exists a point $x^* \in X$ such that for any orthogonal element $x_0 \in X$, the iteration sequence $\{f^n x_0\}$ converges to this point. Also, if f is $\perp$-continuous at $x^* \in X$, then $x^* \in X$ is a unique fixed point of f.

If we assume $\phi(t) = t$ in Corollary 12, we get the following corollary:

Corollary 13 *Let $(X, \perp, d)$ be an O-complete metric space(not necessarily complete metric space),$f : X \to X$ a self map and $\alpha : \mathbb{R}^+ - \{0\} \to [0, 1)$ be decreasing function such that $\alpha(t) < 1$ for every $t > 0$. Suppose that f is $\perp$-preserving self mapping satisfying the inequality*

$$d(fx, fy) \leq \alpha(d(x, y))d(x, y) \tag{25}$$

for all $x, y \in X$ where $x \perp y$ and $x \neq y$. In this case, there exists a point $x^* \in X$ such that for any orthogonal element $x_0 \in X$, the iteration sequence $\{f^n x_0\}$ converges to this point. Also, if f is $\perp$-continuous at $x^* \in X$, then $x^* \in X$ is a unique fixed point of f.

Acknowledgment

The authors would like to thank the referees for their careful reading and their valuable comments and suggestions to improve their results.

REFERENCES

[1] Eshaghi Gordji, M. and Habibi, H. (2017). Fixed point theory in generalized orthogonal metric space. Journal of Linear and Topological Algebra (JLTA), 6(3), 251-260.

[2] Gordji, M. E., Ramezani, M., De La Sen, M. and Cho, Y. J. (2017). On orthogonal sets and Banach fixed point theorem. Fixed Point Theory (FPT), 18(2), 569-578.

[3] Khan, M. S., Swaleh, M. and Sessa, S. (1984). Fixed point theorems by altering distances between the points. Bulletin of the Australian Mathematical Society (Bulletin of the AMS), 30(1), 1-9.