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Communication over Hypercomplex Kähler Manifolds: Capacity of Dual-Polarized Multidimensional-MIMO Channels

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Abstract

We consider single-user communications over joint space-pattern-polarization diversity providing dual-polarized multidimensional-MIMO (MD-MIMO) channels established by the use of multiple dual-polarized transmit/receive antennas in the form of uniformly-spaced 1D, 2D and/or 3D MIMO arrays. Based on the equivalent channel-models formulated based on hypercomplex Kähler manifolds, we subsequently identify the decomposition of dual-polarized multidimensional-MIMO channels into multiple independently-fading and scaled classical MIMO channels in parallel through the algebraic properties of hyperKähler manifolds and consequently derive the corresponding ergodic capacities analytically. We show in essence via the diversity-reception over independent channels perspective deduction of the decomposition into parallel MIMO channels observation that the capacity gains achievable by dual-polarized multidimensional-MIMO Tx/Rx over classical single-polarized linear antenna array MIMO Tx/Rx may be notably large with equal number of transmit and/or receive antenna locuses constrained compactly and under same resource requirements/channel conditions whenever the cross-polar discrimination is good, i.e. $\rho \ll 1$.

1. INTRODUCTION

Multiple-input multiple-output (MIMO) communication techniques with transmit/receive antenna-arrays are receiving ever growing attention for use in concurrent communication systems due to enhanced capacity/quality achievable with respect to single-antenna systems via provision of spatial diversity and multiplexing especially over fading wireless links. Substantially enhanced performance of MIMO systems over single-antenna systems have first been shown by the classical demonstration of Foschini and Gans [1] and the 1999 paper of Telatar followed by other subsequent contributions in quite large quantity on various aspects of MIMO transmission/reception.

In retrospect to the literature, the capacity analysis of MIMO channels has been constrained mainly to the use of linear antenna-arrays of size t_l and r_l at the transmitter and receiver respectively that forms, as we will denote throughout the sequel, a classical (t_l, r_l) single-polarized linear antenna-array MIMO channel. Only a few number of works considered using multiple antennas either in a densely-spaced nonuniform configuration over multiple-dimensions with limitations on the size of the placement region [3] or in the form of fixed-length linear antenna-arrays as array sizes get large [4] to analyze the effects of physical constraints on the capacity of MIMO systems due to inherently strict space-consumption considerations for deployment especially at mobile stations/handheld terminals.

From another perspective, the design of antenna structures for MIMO systems can also be viewed as the design of low correlation diversity channels in a compact/small-sized form. This idea of compact diversity MIMO systems originates from the energy density antenna definition [5] which considers the electric field above a ground plane and the two horizantal magnetic fields yielding three independent signals using a vertical dipole and two horizantal magnetic dipoles. Similarly, the idea to use both space and a combination of polarization and pattern diversity to enhance the decorrelation between diversity channels is also recently used and analyzed in [6-8].

Instead in this sequel, through another perspective

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in terms of rather different system model formulations and analysis approaches than [3-8], we investigate the ergodic capacity for single-user communications over joint space-pattern-polarization diversity providing generalized $(t_1 \times t_2 \times t_3 t_M, r_1 \times r_2 \times r_3 r_M)$ dual-polarized multidimensional-MIMO (PMD-MIMO) channels established by the placement of dual-polarized multiple transmit/receive antennas with uniform interelement spacing on the horizantal/vertical axes of the established grid in the form of 1D, 2D and/or 3D MIMO arrays.

2. PRELIMINARIES AND SYTEM MODEL

For the generalized $(t_1 \times t_2 \times t_3 t_M, r_1 \times r_2 \times r_3 r_M)$ dual-polarized PMD-MIMO channel that we mainly investigate throughout the sequel placed with uniform interelement (dual-polarized antenna) spacing in the form of 1D, 2D and/or 3D MIMO arrays composed of $2t_M \times 2r_M$ single-path links and with total transmitter power-constraint on the codeword-letters P_M, the discrete-time transfer-function matrix in each channel-use has physically structural representation over the complex topological tensor-product space $\left\{ \mathcal{T}^{2n \otimes 2m}_{\otimes_2^{\mathcal{C}}}(\mathcal{C}, l_M) : V \overset{\otimes_2^{\mathcal{C}}}{\rightarrow} W; Dim(V) = p, Dim(W) = q \right\}$ on complex-numbers field $\mathcal{C}$ equipped with the noncommutative topological tensor-product operator $< (.) \otimes_2^{\mathcal{C}} (.) >$ with tensor-dimension $l_M = Dim\left(\overset{\leftrightarrow}{\mathbf{T}}\left(\mathcal{T}^{2n \otimes 2m}_{\otimes_2^{\mathcal{C}}}(\mathcal{C}, l_M)\right)\right) = \max(p, q)$ revealing the order M of the generalized dual-polarized PMD-MIMO channel as; $\mathbf{H}_M[n] \in \mathcal{T}^{2r_M \otimes 2t_M}_{\otimes_2^{\mathcal{C}}}(\mathcal{C}, l_M)$, that yields the following input-output relation channel-model in each channel-use respectively:

$$\mathbf{Y}_M[n] = \mathbf{H}_M[n] \otimes_2^{\mathcal{C}} \mathbf{X}_M[n] + \mathbf{N}_M[n], \forall n \qquad (1)$$

where $\mathbf{X}_M[n] \in \mathcal{C}^{t_1 \times t_2 \times t_3}$ is the array of input codeword-letters, $\mathbf{Y}_M[n] \in \mathcal{C}^{r_1 \times r_2 \times r_3}$ is the array of soft-output sufficient-statistics for joint maximum-likelihood decoding over receive antennas, and $\mathbf{N}_M[n] \in \mathcal{C}^{r_1 \times r_2 \times r_3}$ is the array of spatially-independent elementwise zero-mean unity-variance circularly-symmetric complex background-AWGN samples with variances of σ_M^2 over all receive antennas respectively. It is noteworthy here that each interelement link between dual-polarized antenna pairs at the receiver and transmitter in this channel description is infact topologically described by a 2×2 matrix $\mathbf{H}_M^{(i/j)}[n] = \begin{pmatrix} h_{0,0} & h_{0,1} \\ h_{1,0} & h_{1,1} \end{pmatrix}$ where all 4 elements are zero-mean and uncorrelated, $\mathcal{E}\left\{|h_{0,0}|^2\right\} = \mathcal{E}\left\{|h_{1,1}|^2\right\} = 1$, $\mathcal{E}\left\{|h_{0,1}|^2\right\} = \mathcal{E}\left\{|h_{1,0}|^2\right\} = \rho$. The term ρ is the *cross-polar mixture ratio* and depicts the ratio of power in the cross-polar components to the power in the copolar components.

The transfer-function matrix in generalized dual-polarized PMD-MIMO(M) channel-model defined over $\mathcal{T}^{2n \otimes 2m}_{\otimes_2^{\mathcal{C}}}(\mathcal{C}, l_M)$ in (1) is infact in a rather unfamiliar region mathematically-speaking to further proceed towards any corresponding capacity derivations via linear-algebraic methods, and an equivalent transformation from $\mathcal{T}^{2n \otimes 2m}_{\otimes_2^{\mathcal{C}}}(\mathcal{C}, l_M)$ to a Euclidean-like vector inner-product space will be facilitative. Towards this goal, via the algebraic properties of topological tensor-product spaces as identified by the symplectic-topology branch of mathematics, a complex topological tensor-product space $\left\{ \mathcal{T}^{2n \otimes 2m}_{\otimes_2^{\mathcal{C}}}(\mathcal{C}, l_M) : V \overset{\otimes_2^{\mathcal{C}}}{\rightarrow} W; Dim(V) = p, Dim(W) = q \right\}$ with tensor-dimmension $Dim\left(\overset{\leftrightarrow}{\mathbf{T}}\left(\mathcal{T}^{2n \otimes 2m}_{\otimes_2^{\mathcal{C}}}(\mathcal{C}, l_M)\right)\right) = l_M = \max(p, q)$, the corresponding tensor-rank; $Rank\left(\overset{\leftrightarrow}{\mathbf{T}}\left(\mathcal{T}^{2n \otimes 2m}_{\otimes_2^{\mathcal{C}}}(\mathcal{C}, l_M)\right)\right) = h_M = 2^{l_M}$ and the tensor-valence; $Val\left(\overset{\leftrightarrow}{\mathbf{T}}\left(\mathcal{T}^{2n \otimes 2m}_{\otimes_2^{\mathcal{C}}}(\mathcal{C}, l_M)\right)\right) = (l_M, 0)$ - reference-frame shift-invariant-indexed Cartesian-tensors -, has topologically-algebraic-isomorphism/symplectomorphism onto an $2n \times 2m$ symplectic hypercomplex Kähler - hyperKähler - manifold; viz. topologically-compact/connected locally-Euclidean vector-fields, over h_M-dimensional real-Hamiltonian vector-space $\mathcal{H}$ on $\mathcal{R}^{h_M}$ equipped with the inner-product operator $\odot_{h_M}^{\mathcal{H}}$; i.e. $\mathcal{T}^{2n \otimes 2m}_{\otimes_2^{\mathcal{C}}}(\mathcal{C}, l_M) \overset{\mathcal{I}_T}{\Leftrightarrow} \mathcal{K}_{\odot_{h_M}^{\mathcal{H}}}[\mathcal{H}(\mathcal{R}^{h_M})]^{2n \times 2m}$, where the operator $(.) \overset{\mathcal{I}_T}{\Leftrightarrow} (.)$ stands for algebraic-isomorphism between topological tensor-product spaces (Moser Lemma and Darboux Theorem, Lecture 10,[9]).

Through this key algebraic-isomorphism property, the transfer function matrix $\mathbf{H}_M[n] \in T^{(r_1 \times r_2 \times r_3) \otimes (t_1 \times t_2 \times t_3)}_{\otimes_2^{\mathcal{C}}}(\mathcal{C}, l_M)$ for a generalized dual-polarized PMD-MIMO(M) channel has linear-algebraically equivalent mapping depending on the channel order M onto the three main hyperKähler manifolds respectively that are namely, the one defined over the complex vector-space $\mathcal{C}$ on $\mathcal{R}^2$ for $M = 1$; i.e. $\mathbf{H}_1[n] \in T^{(r_1 \times r_2 \times r_3) \otimes (t_1 \times t_2 \times t_3)}_{\otimes_2^{\mathcal{C}}}(\mathcal{C}, 1) \overset{\mathcal{I}_T}{\Leftrightarrow} \tilde{\mathbf{H}}_1[n] \in \mathcal{K}_{\odot_2^{\mathcal{C}}}[\mathcal{C}(\mathcal{R}^2)]^{2r_M \times 2t_M}$ with the associated complex inner-product operator $\odot_2^{\mathcal{C}}$, the one defined over the double-orthocomplex real-quaternionic-

■	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_1$	$\mathbf{e}_1$	-1	$\mathbf{e}_3$	$\mathbf{e}_2$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_3$	-1	$\mathbf{e}_1$
$\mathbf{e}_3$	$\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_1$	-1

(a)

■	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
$\mathbf{e}_1$	$\mathbf{e}_1$	-1	$\mathbf{e}_4$	$\mathbf{e}_7$	$-\mathbf{e}_2$	$\mathbf{e}_6$	$-\mathbf{e}_5$	$-\mathbf{e}_3$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_4$	-1	$\mathbf{e}_5$	$\mathbf{e}_1$	$-\mathbf{e}_3$	$\mathbf{e}_7$	$-\mathbf{e}_6$
$\mathbf{e}_3$	$\mathbf{e}_3$	$-\mathbf{e}_7$	$-\mathbf{e}_5$	-1	$\mathbf{e}_6$	$\mathbf{e}_2$	$\mathbf{e}_4$	$\mathbf{e}_1$
$\mathbf{e}_4$	$\mathbf{e}_4$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$-\mathbf{e}_6$	-1	$\mathbf{e}_7$	$\mathbf{e}_3$	$-\mathbf{e}_5$
$\mathbf{e}_5$	$\mathbf{e}_5$	$-\mathbf{e}_6$	$\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_7$	-1	$\mathbf{e}_1$	$\mathbf{e}_4$
$\mathbf{e}_6$	$\mathbf{e}_6$	$\mathbf{e}_5$	$-\mathbf{e}_7$	$\mathbf{e}_4$	$-\mathbf{e}_3$	$-\mathbf{e}_1$	-1	$\mathbf{e}_2$
$\mathbf{e}_7$	$\mathbf{e}_7$	$\mathbf{e}_3$	$\mathbf{e}_6$	$-\mathbf{e}_1$	$\mathbf{e}_5$	$-\mathbf{e}_4$	$-\mathbf{e}_2$	-1

(b)

Figure 1: (a): Hamiltonian-rules for noncommutative quaternionic bases multiplication, (b): Cayley-rules for noncommutative octonionic bases multiplication

Hamiltonian vector-space $\mathcal{Q}$ on $\mathcal{R}^4$ for $M = 2$; i.e. $\mathbf{H}_2[n] \in T_{\otimes_2^{\mathcal{C}}}^{(r_1\times r_2\times r_3)\otimes(t_1\times t_2\times t_3)}(\mathcal{C},2) \overset{\mathcal{I}_T}{\Leftrightarrow} \tilde{\mathbf{H}}_2[n] \in \mathcal{K}_{\odot_4^{\mathcal{Q}}}[\mathcal{Q}(\mathcal{R}^4)]^{2r_M\times 2t_M}$ with the associated real-quaternionic inner-product operator $\odot_4^{\mathcal{Q}}$ and the one defined over quadruple-orthocomplex real-octonionic-Hamiltonian vector-space $\mathcal{O}$ on $\mathcal{R}^8$ for $M = 3$; $\mathbf{H}_3[n] \in T_{\otimes_2^{\mathcal{C}}}^{(r_1\times r_2\times r_3)\otimes(t_1\times t_2\times t_3)}(\mathcal{C},3) \overset{\mathcal{I}_T}{\Leftrightarrow} \tilde{\mathbf{H}}_3[n] \in \mathcal{K}_{\odot_8^{\mathcal{O}}}[\mathcal{O}(\mathcal{R}^8)]^{2r_M\times 2t_M}$ with the associated real-octonionic inner-product operator $\odot_8^{\mathcal{O}}$. Hence, the input-output relation channel-model for generalized dual-polarized PMD-MIMO(M) channels in (1) admits the following linear-algebraically familiar-ground equivalent representation in each channel-use respectively:

$$\tilde{\mathbf{y}}_M[n] = \tilde{\mathbf{H}}_M[n]\Xi_{h_M}^{\mathcal{H}}\tilde{\mathbf{x}}_M[n] + \tilde{\mathbf{w}}_M[n], \forall n \qquad (2)$$

where $\tilde{\mathbf{x}}_M[n] \in \mathcal{H}^{2t_M\times 1}$ is the vector of input codeword-letters, $\tilde{\mathbf{y}}_M[n] \in \mathcal{H}^{2r_M\times 1}$ is the soft-output sufficient-statistics vector, and $\tilde{\mathbf{w}}_M[n] \in \mathcal{H}^{2r_M\times 1}$ is the background-AWGN noise vectors with variances of $\sigma_M^2/2$ on each basis.

Some basic facts on real-quaternions and real-octonions invented by W. R. Hamilton in 1846 [10] relevant hereinforth to our investigation are that they are hypercomplex generalizations of complex numbers over unit-norm 4-tuple-basis vector-space $\left\{1, \{\mathbf{e}_i\}_{i=1}^3\right\}$; i.e. $\mathbf{q} = a_0 + \sum_{i=1}^3 a_i\mathbf{e}_i \in \mathcal{Q}$ with quaternionic-conjugate $\mathbf{q}^{*_4^{\mathcal{Q}}} = a_0 - \sum_{i=1}^3 a_i\mathbf{e}_i$, and unit-norm 8-tuple-basis vector-space $\left\{1, \{\mathbf{e}_i\}_{i=1}^7\right\}$; $\mathbf{o} = b_0 + \sum_{i=1}^3 b_i\mathbf{e}_i \in \mathcal{O}$ with octonionic-conjugate $\mathbf{o}^{*_4^{\mathcal{O}}} = b_0 - \sum_{i=1}^3 b_i\mathbf{e}_i$, with the identity element $\{1\}$ that both forms a noncommutative division-algebra on $\mathcal{R}^4$ and $\mathcal{R}^8$ respectively. For the definition of real-quaternionic matrix/vector multiply $\Xi_4^{\mathcal{Q}}$ in terms of inner-product $\odot_4^{\mathcal{Q}}$ and real-octonionic matrix/vector multiply $\Xi_8^{\mathcal{O}}$ in terms of inner-product $\odot_8^{\mathcal{O}}$, the addition of both quternions and octonions are elementwise between each basis as with the complex numbers, while the multiplication of quaternion and octonion bases obey the noncommutative rules respectively given by the *Hamiltonian-rules* defined in terms of noncommutative 3-element circle and the *Cayley-rules* defined in terms of noncommutative 7-element Fano projective-plane that are both given as tabulated in Fig. 1.

3. ERGODIC CAPACITIES

For the derivation of the ergodic capacity for generalized dual-polarized PMD-MIMO(M) channels under receiver-side complete CSI only and no transmitter-CSI, it is still optimal to uniformly distribute the total transmitter power over all transmit antennas as conjecturized by Telatar in [2] for the classical MIMO channel due to our assumption of statistically-i.i.d. fading over all interelement links and the ergodic capacity definition via the equivalent channel-model of (2) in nats/sec/Hz is given by:

$$C_e^{(M)} = \mathcal{E}_{\tilde{\mathbf{H}}_M\Xi_{h_M}^{\mathcal{H}}\tilde{\mathbf{H}}_M^{\dagger}}\left\{\ln(|\mathbf{K}_{\tilde{\mathbf{w}}_M} + \frac{P_M}{t_M}\tilde{\mathbf{H}}_M\Xi_{h_M}^{\mathcal{H}}\tilde{\mathbf{H}}_M^{\dagger_{h_m}^{\mathcal{H}}}|)\right\} \qquad (3)$$

where $\mathbf{K}_{\tilde{\mathbf{w}}_M} = \mathcal{E}\{\tilde{\mathbf{w}}_M \odot_{h_M}^{\mathcal{H}} \tilde{\mathbf{w}}_M^{\dagger_{h_m}^{\mathcal{H}}}\}$ is the covariance matrix of complex, quaternionic and octonionic background-AWGN vectors $\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_2$ and $\tilde{\mathbf{w}}_3$ with $(.)^{\dagger_{h_M}^{\mathcal{H}}}$ standing for the complex, quaternionic and octonionic Hamiltonian-Hermitian operators $(.)^{\dagger_2^{\mathcal{C}}}, (.)^{\dagger_4^{\mathcal{Q}}}$ and $(.)^{\dagger_8^{\mathcal{O}}}$ in terms of the quaternionic and octonionic conjugate operators $(.)^{*_2^{\mathcal{C}}}, (.)^{*_4^{\mathcal{Q}}}$ and $(.)^{*_8^{\mathcal{O}}}$ for dual-polarized PMD-MIMO(1), PMD-MIMO(2) and PMD-MIMO(3)

respectively.

Remembering via the equivalent channel-model of a generalized PMD-MIMO(M) channel in (2) that; $\tilde{\mathbf{H}}_M = \mathbf{H}_M(0).1 + \sum_{i=1}^{2^M-1} \mathbf{H}_M(i).\mathbf{e}_i$ with $\{\mathbf{H}_M(i)\}_{i=0}^{2^M-1}$ being mutually statistically-i.i.d. $2r_M \times 2t_M$ random matrices, a direct observation from the noncommutative Hamiltonian and Cayley tables in Fig. 1 as well as the properties of complex numbers modularly yield the following reduced equivalent form after some algebra:

$$\tilde{\mathbf{W}}_M = \tilde{\mathbf{H}}_M \Xi_{h_m}^{\mathcal{H}} \tilde{\mathbf{H}}_M^{\dagger_{h_M}^{\mathcal{H}}} \overset{D_\lambda}{\equiv} \sum_{i=0}^{2^M-1} \mathbf{H}_M(i)\mathbf{H}_M^T(i) \overset{D_\lambda}{\equiv} \sum_{i=1}^{2^{M-1}} \left(\frac{1-\sqrt{\rho}}{1-\rho}\right)^2 \mathbf{W}_M(i) \quad (4)$$

where $\{\mathbf{W}_M(i)\}_{i=1}^{2^{M-1}}$ are $\min(2r_M, 2t_M)$-rank mutually statistically-i.i.d. random matrices in complex-Wishart ensembles $\mathcal{WH}(\max(2r_M, 2t_M), \min(2r_M, 2t_M), 1)$ and the operator ($\overset{D_\lambda}{\equiv}$) stands for the eigenvalue distribution-equivalence between random matrices; i.e. $\mathbf{A} \overset{D_\lambda}{\equiv} \mathbf{B} \leftrightarrow f_{\mathbf{A}}(\lambda) = f_{\mathbf{B}}(\lambda)$.

Via the very same observation, the noise covariance matrix $\mathbf{K}_{\tilde{\mathbf{w}}_M}$ of a generalized dual-polarized PMD-MIMO(M) channel, where $\tilde{\mathbf{w}}_M = \mathbf{w}_M(0).1 + \sum_{i=1}^{2^M-1} \mathbf{w}_M(i).\mathbf{e}_i$ with $\{\mathbf{w}_M(i)\}_{i=0}^{2^M-1}$ being mutually statistically-i.i.d. $2r_M \times 1$ real-Gaussian random vectors composed of mutually-i.i.d. zero-mean $\frac{\sigma_M^2}{2}$-variance elements, also further admits the following form respectively:

$$\mathbf{K}_{\tilde{\mathbf{w}}_M} = \mathcal{E}\left\{\sum_{i=0}^{2^M-1} \mathbf{w}_M(i)\mathbf{w}_M^T(i)\right\} = 2^{M-1}\sigma_M^2 \mathbf{I}_{2r_M} \quad (5)$$

through which the following hidden aspect is infact brought out to daylight in conjunction with (4):

Proposition 1.: A flat Rayleigh-fading generalized dual-polarized ($t_1 \times t_2 \times t_3 t_M, r_1 \times r_2 \times r_3 r_M$) PMD-MIMO($M$) channel with noise variance per transmit/receive polarized antennas σ_M^2 and cross-polar mixture ratio ρ is equivalent to 2^{M-1} $(2t_M, 2r_M)$ independently-fading classical MIMO channels in parallel each composed of $m_M = \min(2t_M, 2r_M)$ parallel SISO subchannels scaled with $\left(\frac{1-\sqrt{\rho}}{1-\rho}\right)^2$ and with equivalent noise variance per subchannel per parallel MIMO channel σ_M^2.

Hence, the ergodic capacity expression in (3) can be written equivalently via (4) and (5) in terms of the eigenvalues of $\tilde{\mathbf{W}}_M = \sum_{i=1}^{2^{M-1}} \mathbf{W}_M(i)$ by successively decomposing determinant into the products of the eigenvalues and using the log-product identity as:

$$C_e^{(M)} = \mathcal{E}\left\{\sum_{k=1}^{m_M} \ln\left(1 + \left(\frac{1-\sqrt{\rho}}{1-\rho}\right)^2 \bar{\gamma}_s^{(M)} \lambda_k(\tilde{\mathbf{W}}_M)\right)\right\} \quad (6)$$

where $\bar{\gamma}_s^{(M)} = \frac{1}{2t_M}\frac{P_M}{\sigma_M^2} = \frac{1}{2t_M}\frac{E_s^{(M)}}{N_0^{(M)}}$ is the average received symbol SNR per antenna for total transmitter symbol SNR $\gamma_s^{(M)} = \frac{P_M}{\sigma_M^2}$, symbol energy $E_s^{(M)}$ and noise PSD level $N_0^{(M)}$.

To analytically evaluate the ergodic capacity expression for generalized dual-polarized MD-MIMO(M) channels in (3) via the equivalent decomposition in (6), the analytic form of the eigenvalue density for $\tilde{\mathbf{W}}_M$ is further required. $\tilde{\mathbf{W}}_M$ is a random matrix in the form of the sums of 2^{M-1} mutually statistically-i.i.d. $\min(2t_M, 2r_M)$-rank random matrices $\mathbf{W}_M(i)$ each within complex-Wishart ensembles $\mathcal{WH}(\max(2t_M, 2r_M), \min(2t_M, 2r_M), 1)$. The eigenvalue density of an $n \times m$ complex-Wishart random matrix with common variance of unity for finite n and m parameters is known to obey *complex-Wishart density* [Theorem 5.4, 11] with the analytic form:

$$f_{\mathcal{WH}(n,m,1)}(\lambda) = \frac{1}{m}\sum_{k=0}^{m-1} \left[\psi_k^{n-m}(\lambda)\right]^2,\ \lambda \geq 0 \quad (7)$$

in terms of the sequence of Laguerre functions; $\psi_k^\alpha(x) = \left[\frac{k!}{(k+\alpha)!}x^\alpha e^{-x}\right]^{\frac{1}{2}} L_k^\alpha(x)$, $k \in \mathcal{N}^+, x \geq 0$; where $L_k^\alpha(x) = \frac{1}{k!}x^{-\alpha}e^x\frac{d^k}{dx^k}\left[x^{k+\alpha}e^{-x}\right]$, $k \in \mathcal{N}^+, x \geq 0$, is the αth order kth Laguerre polynomial [12]. Furthermore, the following functional averaging is known to apply for complex-Wishart Ensembles [Corollary 5.6, 11]:

$$\mathcal{E}\left\{\frac{1}{m}\sum_{i=1}^{m} g\left(\lambda_k(\mathbf{W})\right)\right\} = \int_0^\infty g(\lambda) f_{\mathcal{WH}(n,m,1)}(\lambda) d\lambda \quad (8)$$

for an analytic function $g(x) \geq 0$. Thus, via the usual *independence-and-convolution property* [13] of classical probability theory for the finite array sizes we consider, the analytic eigenvalue density of $\tilde{\mathbf{W}}_M$ is given by:

$$f_{\tilde{\mathbf{W}}_M}(\lambda) =_{2^{M-1}} f_{\mathcal{WH}(n_M, m_M, 1)}(\lambda) \quad (9)$$

where $n_M = \max(2t_M, 2r_M)$, $m_M = \min(2t_M, 2r_M)$, and the operator ${}_K f(x)$ stands for the $(K-1)$-fold standard convolution of the function $f(x)$ with itself. Furthermore, by functional averaging, the ergodic capacity expression in (6) is equal to:

$$C_e^{(M)} = m_M \int_0^\infty \ln\left(1 + \left(\frac{1-\sqrt{\rho}}{1-\rho}\right)^2 \bar{\gamma}_s^{(M)} \lambda\right) f_{\tilde{\mathbf{W}}_M}(\lambda) d\lambda \quad (10)$$

through which we analytically formulate the ergodic capacity for generalized dual-polarized PMD-MIMO(M) channels modularly in the following theorem as an implicit integral of 2^{M-1}-fold sums of convolutive squared Laguerre functions:

Theorem 1.: The ergodic capacity of an $(t_1 \times t_2 \times t_3 t_M, r_1 \times r_2 \times r_3 r_M)$ flat Rayleigh-fading generalized dual-polarized PMD-MIMO(M) channel in nats/sec/Hz for $M = 1, 2, 3$, for finite but arbitrary t_M and r_M and cross-polar mixture ratio between dual-polarized antennas ρ, under receiver-side complete CSI and no transmitter-CSI; hence total-transmitter power P_M uniformly-distributed over all transmit antennas, is given by:

$$\begin{aligned} C_e^{(M)} &= \frac{1}{m_M 2^{2M-1-1}} \int_0^\infty \ln\left(1 + \left(\frac{1-\sqrt{\rho}}{1-\rho}\right)^2 \bar{\gamma}_s^{(M)} \lambda\right) \\ &\times \sum_{k_1=0}^{m_M-1} \cdots \sum_{k_{2^{M-1}}=0}^{m_M-1} \\ &\left\{ \begin{array}{c} \left[\psi_{k_1}^{n_M-m_M}(\lambda)\right]^2 * \left[\psi_{k_2}^{n_M-m_M}(\lambda)\right]^2 \cdots \\ * \left[\psi_{k_{2^{M-1}}}^{n_M-m_M}(\lambda)\right]^2 \end{array} \right\} d\lambda \end{aligned} \quad (11)$$

where $n_M = \max(2t_M, 2r_M)$, $m_M = \min(2t_M, 2r_M)$, $\bar{\gamma}_s^{(M)} = \frac{1}{2t_M}\frac{P_M}{\sigma_M^2} = \frac{1}{2t_M}\frac{E_s^{(M)}}{N_0^{(M)}}$ is the average received symbol SNR per antenna for total transmitter symbol SNR $\gamma_s^{(M)} = \frac{P_M}{\sigma_M^2}$, $E_s^{(M)}$ is the symbol energy, $N_0^{(M)}$ is the noise PSD level. The operator $< (.) * (.) >$ stands for the standard convolution, and $\psi_k^\alpha(x) = \left[\frac{k!}{(k+\alpha)!} x^\alpha e^{-x}\right]^{\frac{1}{2}} L_k^\alpha(x)$, $k \in \mathcal{N}^+$, $x \geq 0$; is the αth-order kth Laguerre function in terms of the αth-order kth Laguerre polynomial $L_k^\alpha(x) = \frac{1}{k!} x^{-\alpha} e^x \frac{d^k}{dx^k}\left[x^{k+\alpha} e^{-x}\right]$, $k \in \mathcal{N}^+$, $x \geq 0$.

4. NUMERICAL RESULTS AND DISCUSSION

Via Proposition 1., a direct intuitive insight from diversity-reception over independent channels perspective gives the vital hint on substantial capacity gains achievable by dual-polarized PMD-MIMO Tx/Rx over classical MIMO Tx/Rx with equal number of transmit and/or receive antenna locuses constrained compactly and under same resource requirements/channel conditions, i.e. equal total transmitter symbol SNR $\gamma_s^{(M)}$ and channel coefficient average powers.

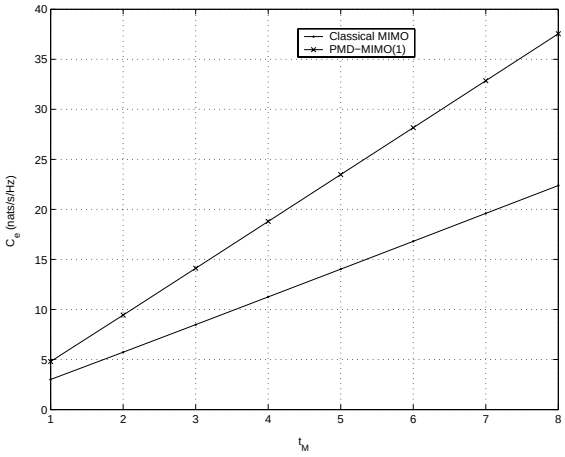


Figure 2: Ergodic capacity of classical MIMO and PMD-MIMO(1) channels versus t_M for $t_M = r_M$ case with total transmit symbol SNR of 15 dB and cross-polar discrimination 10 dB.

Representative examples of the capacity gains under ergodic transmission scenarios achievable by dual-polarized PMD-MIMO Tx/Rx over MIMO Tx/Rx via analytic expression (11) at $t_M = r_M$ versus t_M for classical MIMO and PMD-MIMO(1) are presented in Fig. 2 and versus cross polar discrimination in dB at $t_M = r_M = 4$ for classical MIMO and PMD-MIMO(2) parametrized by total transmitter symbol SNR $\gamma_s^{(M)}$ in dB Fig. 3. It turns out via the analysis of the case presented in Fig. 2 that the capacity gain achievable by PMD-MIMO(1)Tx/Rx over classical MIMO Tx/Rx scales up linearly with the number of antenna locuses used. Furthermore, via Fig. 2 where we present the detrimental impact of high cross-polar mixture on the ergodic capacity of PMD-MIMO(2), it further turns out that the adverse affect of high crosspolar mixture on the capacity gain of PMD-MIMO(2) Tx/Rx with respect to classical MIMO Tx/Rx is reduced with increased total transmitter symbol SNR $\gamma_s^{(M)}$.

5. CONCLUSION

We analytically evaluated and analyzed the er-

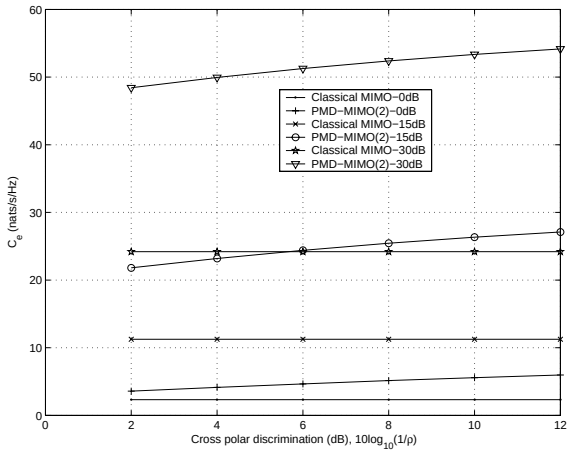


Figure 3: Ergodic capacity of classical MIMO and PMD-MIMO(2) channels for $t_M = t_M = 4$ versus cross-polar discrimination in dB parametrized by total transmit symbol SNR in dB.

godic capacity for single-user communications over generalized dual-polarized multidimensional-MIMO channels. Based on the decomposition of dual-polarized multidimensional-MIMO channels into independently-fading and scaled parallel classical MIMO channels observation, the capacity gains for ergodic scenarios achievable by joint space-pattern-polarization diversity providing dual-polarized multidimensional-MIMO Tx/Rx over classical MIMO Tx/Rx with equal number of transmit and/or receive antennas locuses constrained compactly and under same resource requirements/channel conditions is presented whenever the cross-polar discrimination between dual antennas is good; i.e. $\rho \ll 1$.

REFERENCES

[1] G. J. Foschini and M. J. Gans. On the Limits of Wireless Communications in a Fading Environment When Using Multiple Antennas. Wireless Personal Communications 6: 311-335, 1998.

[2] I. E. Telatar. Capacity of Multi-antenna Gaussian Channels. European Transactions on Telecommunications, vol. 10, no. 6, pp. 585-595, Nov./Dec. 1999.

[3] N. Chiurtu, B. Rimoldi, E. Telatar and V. Pauli. On the Capacity of Spatially-Dense Multiple Antenna Systems. Proceedings of Winter School on Coding and Information Theory, Monte Verita, Switzerland, February 24-27 2003.

[4] W. Shuanqing, D. L. Goeckel and R. Janaswamy. On the Asymptotic Capacity of MIMO Systems with Fixed Length Linear Antenna Arrays. Proceedings of IEEE International Conference on Communications, 2003, vol. 4, pp. 2633 - 2637, May 2003.

[5] W. C. Y. Lee. *Mobile Communications Engineering.* McGraw-Hill, New York, 1982.

[6] M. A. Andrews, P. P. Mitra and R. Carvalho. Tripling the Capacity of Wireless Communications using Electromagnetic Polarization. Nature, vol. 409, pp. 316-318, January 2001.

[7] T. Svantesson. On the Capacity and Correlation of Multi-Antenna Systems employing Multiple Polarizations. Proceedings of IEEE International Symposium on Antennas and Propagation Society, vol. 3, pp. 202-205, San Antonio, Texas, June 2002.

[8] Beza Negash Getu and Jorgen Bach Andersen. The MIMO Cube - A Compact MIMO Antenna. IEEE Transactions on Wireless Communications, vol. 4, no. 3, pp. 1136-1141, May 2005.

[9] Yves Talpaert. *Differential Geometry with Applications to Mechanics and Physics.* Marcel Drekker Inc., New York, 2001.

[10] Ian R. Porteus. *Clifford Algebras and the Classical Groups* . Press Syndicate of the University of Cambridge, Canbridge, 1995.

[11] U. Haagerup and S. Thorbjornsen. Random Matrices with Complex Gaussian Entries. Pre-print, Odense University, 1998.

[12] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series and Products.* San Diego, CA: Academic Press, 5th edition, 1994.

[13] A. Papoulis. *Probability, Random Variables and Stochastic Processes.* McGraw-Hill Inc. Press, 3rd edition, 1991.