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Far-field patterns for time-harmonic acoustic plane waves in an inhomogeneous medium

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Abstract

This paper is concerned with the class of far-field patterns corresponding to the scattering of time-harmonic acoustic plane waves by an inhomogeneous medium which have $\rho_i(x) > 0$ continuously differentiable density. This class is shown to be complete in $L^2(\partial\Omega)$ (where $\partial\Omega$ is the unit sphere) under the same restriction. The existence of a unique weak solution to the interior transmission problem is also established. © 2000 Elsevier Science Inc. All rights reserved.

1. Introduction

Recently, for a constant density ρ_i, several methods have been given [1–3,6,7,9,11,12] for constructing a numerical solution of the inverse scattering problem for a given $F(\hat{x}; k, \alpha)$ far-field pattern in both two and three dimensions. In Ref. [2] it is shown that this class of far-field patterns is complete in $L^2(\partial\Omega)$ provided that the wave number is sufficiently small. The purpose of this paper is to show that when the class of far-field patterns corresponding to the scattering of time-harmonic plane waves by an inhomogeneous medium is not all of $L^2(\partial\Omega)$. We use here the procedure used by Colton [2], Colton and Monk [6,7], Colton and Kirsch [3] and Kirsch [9]. So we first consider the class of far-field patterns corresponding to plane waves propogating in arbitrary directions which are incident upon a penetrable scattering medium in $\mathbb{R}^3$. Second we use a projection theorem which decomposes the spaces of square integrable functions

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$L^2(\partial\Omega)$ into the direct sum subspaces determined by the class of far-field patterns and the orthogonal complement of these subspaces. Third we get some results for the class of far-field patterns complete in $L^2(\partial\Omega)$. Finally in the last section we prove the uniqueness of the solution of the inhomogeneous interior transmission problem.

2. Far-field patterns for acoustic waves in an inhomogeneous medium

Consider the propogation of a time-harmonic plane wave $u^i(x) = \exp(ikx \cdot \alpha)$ through a simply connected inhomogeneous medium $D \subset \mathbb{R}^3$ of compact support, where the boundary ∂D of D is twice continuously differentiable with unit outward normal n, ln D. Let $a \geqslant 0$ denote the absorption coefficient, $c(x) > 0$, $\rho_i(x) > 0$, $x \in \overline{D}$ the continuously differentiable functions that are local speed of sound and local density respectively. Assume that in the (non-absorbing) host medium $\mathbb{R}^3\backslash\overline{D}$ the density is $\rho_e > 0$ and speed of sound is $c_o > 0$, where ρ_e and c_o are constants. Then if $\omega > 0$ is the frequency and α, $|\alpha| = 1$ is the direction of propogation of the incident plane wave, $k = \omega/c_o$ is the wave number, $\kappa^2 = \omega(\omega + ia)/c_o^2$ and $n(x) = (c_o/c(x))^2$ [6,7]. The mathematical problem to be solved is to determine functions $u_i \in C^2(D) \cap C^1(\overline{D})$, $u_e \in C^2(\mathbb{R}^3\backslash D) \cap C^1(\mathbb{R}^3\backslash\overline{D})$ such that

$$\begin{aligned} &\Delta u_e + k^2 u_e = 0 \text{ in } \mathbb{R}^3\backslash\overline{D}, \\ &\Delta u_i + \kappa^2 n(x) u_i = 0 \text{ in } D, \end{aligned} \tag{2.1}$$

$$\begin{aligned} &\zeta_e u_e - \zeta_i(x) u_i(x) = 0 \text{ on } \partial D, \\ &\frac{\partial u_e}{\partial n} - \frac{\partial u_i}{\partial n} = 0 \text{ on } \partial D, \end{aligned} \tag{2.2}$$

where $u_e(x) = \exp(ikx \cdot \alpha) + u^s(x)$ and the scattered field u^s satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - ik_e u^s\right) = 0 \tag{2.3}$$

uniformly in all directions $\hat{x} = x/|x|$. In the above equations, $r = |x|$, $\zeta_i(x) = (a - i\omega)\rho_i(x)$, $\zeta_e = -i\omega\rho_e$.

Let

$$\Phi(x, y) = -\frac{1}{4\pi}\frac{e^{ik|x-y}}{|x-y|}, \quad x \neq y, \tag{2.4}$$

be the fundamental solution to the Helmholtz equation $\Delta u + k^2 u = 0$. If we apply Green's second identity for $u^s(x)$ and $\Phi(x, y)$, we have

$$u^s(x) = -\int_{\partial D}\left[u^s(y)\frac{\partial \Phi(x,y)}{\partial n(y)} - \Phi(x,y)\frac{\partial u^s(y)}{\partial n}\right]\mathrm{d}s(y), \quad x \in \mathbb{R}^3\backslash\overline{D}. \tag{2.5}$$

Similary applying Green's theorem to the functions $u^i(x)$ and $\Phi(x,y)$, we have

$$0 = -\int_{\partial D}\left[u^i(y)\frac{\partial \Phi(x,y)}{\partial n(y)} - \Phi(x,y)\frac{\partial u^i(y)}{\partial n}\right]\mathrm{d}s(y), \quad x \in \mathbb{R}^3\backslash\overline{D}. \tag{2.6}$$

Adding Eqs. (2.5) and (2.6) we obtain for $x \in \mathbb{R}^3\backslash\overline{D}$

$$u^s(x) = -\int_{\partial D}\left[u_e(y)\frac{\partial \Phi(x,y)}{\partial n(y)} - \Phi(x,y)\frac{\partial u_e(y)}{\partial n}\right]\mathrm{d}s(y). \tag{2.7}$$

In Ref. ([5], p. 278) we have

$$\begin{aligned}\frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{|x-y|} &= \frac{\mathrm{e}^{\mathrm{i}k|x|}}{|x|}\left\{\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y} + \mathrm{o}\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty, \frac{\partial}{\partial n(y)}\frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{|x-y|} \\ &= \frac{\mathrm{e}^{\mathrm{i}k|x|}}{|x|}\left\{\frac{\partial}{\partial n(y)}\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y} + \mathrm{o}\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty\end{aligned} \tag{2.8}$$

uniformly for all $y \in \partial D$. Inserting this into Eq. (2.7) we obtain

$$u^s(x) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{|x|}\left\{\frac{1}{4\pi}\int_{\partial D}\left[u_e(y)\frac{\partial \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}}{\partial n(y)} - \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\frac{\partial u_e(y)}{\partial n}\right]\mathrm{d}s(y)\right\} + \mathrm{o}\left(\frac{1}{|x|^2}\right). \tag{2.9}$$

If we define

$$F(\hat{x};k,\alpha) = \frac{1}{4\pi}\int_{\partial D}\left[u_e(y)\frac{\partial \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}}{\partial n(y)} - \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\frac{\partial u_e(y)}{\partial n}\right]\mathrm{d}s(y), \tag{2.10}$$

then u^s takes the form

$$u^s(x) = \frac{\mathrm{e}^{\mathrm{i}kr}}{r}F(\hat{x};k,\alpha) + \mathrm{o}\left(\frac{1}{r^2}\right), \quad r \to \infty. \tag{2.11}$$

The function $F(\hat{x};k,\alpha)$ defined by Eq. (2.10) is called the "far-field pattern" of the scattered field corresponding to the incident field $u^i(x) = \exp(\mathrm{i}kx\cdot\alpha)$. The inverse transmission problem is to determine ∂D and $n(x)$ from a knowledge of F for $P = \{\omega_i : \omega_i \text{ different frequencies}\}$, $Q = \{\alpha_i : \alpha_i \text{ different directions}\}$ and $\hat{x} \in \partial\Omega = \{x \in \mathbb{R}^3 : |x| = 1\}$.

Let $g \in L^2(\partial\Omega)$. Then

$$\int_{\partial\Omega} F(\hat{x};k,\alpha)\,\overline{g(\hat{x})}\,\mathrm{d}s(\hat{x}) = \frac{1}{4\pi}\int_{\partial D}\left[u_e(y)\frac{\partial\overline{V(y;k)}}{\partial n(y)} - \overline{V(y;k)}\frac{\partial u_e(y)}{\partial n}\right]\mathrm{d}s(y), \tag{2.12}$$

where

$$V(y;k) := \int_{\partial\Omega} g(\hat{x})\,\mathrm{e}^{\mathrm{i}k\hat{x}\cdot y}\mathrm{d}s(\hat{x}). \tag{2.13}$$

The function $V(y;k)$ is a solution of the Helmholtz equation $\Delta V + k^2 V = 0$ in $\mathbb{R}^3$. $V(y;k)$ is known as the Herglotz wave function and g is called the Herglotz kernel of V.

In a non-absorbing medium (2.12) takes the form

$$\int_{\partial\Omega} F(\hat{x};k,\alpha)\overline{g(\hat{x})}\,\mathrm{d}s(\hat{x}) = \frac{1}{4\pi\rho_e}\int_{\partial D}\left[\rho_i(y)u_i(y)\frac{\partial\overline{V(y;k)}}{\partial n(y)} - \rho_e\overline{V(y;k)}\frac{\partial u_i(y)}{\partial n}\right]\mathrm{d}s(y). \tag{2.14}$$

Consider the set of $\mathfrak{I}$ of all far-field patterns $F(\hat{x};k,\alpha)$ corresponding to a fixed domain and a countable number of distinct incoming directions α_n; $n = 1,2,3,\ldots$ i.e.,

$$\mathfrak{I} = \{F(\hat{x};k,\alpha_n) : n = 1,2,3,\ldots\}. \tag{2.15}$$

Then considering the elements of $\mathfrak{I}$ as functions of $\hat{x}$, $\mathfrak{I}$ is a subset of $L^2(\partial\Omega)$ and

$$L^2(\partial\Omega) = \overline{\operatorname{span}\,\mathfrak{I}} \oplus \mathfrak{I}^{\perp}, \tag{2.16}$$

where $\mathfrak{I}^{\perp}$ is the orthogonal complement of $\mathfrak{I}$.

However before considering the inverse transmission problem we want to determine under what condition the set span $\{F(\hat{x};k,\alpha_n) : n = 1,2,3,\ldots\}$ is dense in $L^2(\partial\Omega)$ for fixed k (or ω) and $\{\alpha_n\}$ a dense set of vectors on the unit sphere.

Now consider the interior transmission problem. Suppose that there exists a non-trivial Herglotz wave function V and a function $W \in C^2(D) \cap C^1(\overline{D})$ such that

$$\begin{aligned} &\Delta V + k^2\,V = 0 \text{ in } D,\\ &\Delta W + \kappa^2\,n(x)W = 0 \text{ in } D, \end{aligned} \tag{2.17}$$

$$\begin{aligned}&\zeta_i(x)W = \zeta_e V \text{ on } \partial D,\\ &\frac{\partial W}{\partial n} = \frac{\partial V}{\partial n} \text{ on } \partial D.\end{aligned} \tag{2.18}$$

Then Eq. (2.14) takes the form

$$\int_{\partial\Omega} F(\hat{x};k,\alpha)\,\overline{g(\hat{x})}\,\mathrm{d}s(\hat{x}) = \frac{1}{4\pi\rho_e}\int_{\partial D}\rho_i(y)\left[u_i(y)\frac{\partial\overline{W(y)}}{\partial n} - \overline{W(y)}\frac{\partial u_i(y)}{\partial n}\right]\mathrm{d}s(y). \tag{2.19}$$

If we apply Green's first identity to the right-hand side of Eq. (2.19)

$$\begin{aligned}&\int_{\partial D}\rho_i(y)\left[u_i(y)\frac{\partial\overline{W(y)}}{\partial n} - \overline{W(y)}\frac{\partial u_i(y)}{\partial n}\right]\mathrm{d}s(y)\\ &= \int_D \nabla\rho_i(y)\cdot\left(u_i(y)\,\nabla\overline{W(y)} - \overline{W(y)}\,\nabla u_i(y)\right)\mathrm{d}y.\end{aligned} \tag{2.20}$$

Lemma 2.1. *Suppose that there exists a non-trivial Herglotz wave function V and a function $W \in C^2(D)\cap C^1(\overline{D})$ such that V, W be the solution of the Eqs.* (2.17), (2.18) *and*

$$\int_D \nabla\rho_i(y)\cdot\left(u_i(y)\,\nabla\overline{W(y)} - \overline{W(y)}\,\nabla u_i(y)\right)\mathrm{d}y \neq 0. \tag{2.21}$$

Then the kernel of the Herglotz wave function g is not orthogonal to span $\mathfrak{I}$.

Proof. Let $g \in L^2(\partial\Omega)$. Then we have Eq. (2.19). Eq. (2.21) implies

$$\frac{1}{4\pi\rho_e}\int_{\partial D}\rho_i(y)\left[u_i(y)\frac{\partial\overline{W(y)}}{\partial n} - \overline{W(y)}\frac{\partial u_i(y)}{\partial n}\right]\mathrm{d}s(y) \neq 0.$$

Definition 2.1. If k (or ω) is such that the homogeneous interior transmission problem (2.17), (2.18) has a non-trival solution, then k (or ω) is called the transmission eigenvalue.

Theorem 2.1. *Suppose that*

$$\operatorname{Im}\overline{\zeta_e}\zeta_i(x) \geqslant 0, \quad \operatorname{Im}\overline{\zeta_e\kappa^2}\zeta_i(x) \leqslant 0, \qquad x \in D \tag{2.22}$$

and $W \in C^2(D)\cap C^1(\overline{D})$ such that

$$\operatorname{Im}\left(\overline{\zeta_e}\int_D W(y)\,\nabla\zeta_i(y)\,\cdot\,\nabla\overline{W(y)}\,\mathrm{d}y\right)\geqslant 0. \tag{2.23}$$

Then k (or ω) is not a transmission eigenvalue.

Proof. From Green's formula we have that if V and W are solutions of Eqs. (2.17) and (2.18) then

$$\int_{\partial D} V(y;k)\,\frac{\partial\overline{V(y;k)}}{\partial n(y)}\,\mathrm{d}s(y)=\int_D|\nabla V(y;k)|^2\,\mathrm{d}y-k^2\int_D|V(y;k)|^2\,\mathrm{d}y \tag{2.24}$$

and similarly

$$\begin{aligned}&\int_{\partial D}\zeta_i(y)\,W(y;k)\frac{\partial\overline{W(y;k)}}{\partial n(y)}\,\mathrm{d}s(y)\\&=\int_D\zeta_i(y)\,|\nabla W(y;k)|^2\,\mathrm{d}y\\&\quad+\int_D W(y)\nabla\zeta_i(y)\,\nabla\overline{W(y)}\,\mathrm{d}y-\overline{\kappa^2}\int_D n(y)\zeta_i(y)\,|\nabla W(y;k)|^2\,\mathrm{d}y.\end{aligned} \tag{2.25}$$

If we consider the imaginary part of Eq. (2.24), then we have

$$0=\operatorname{Im}\left(\int_{\partial D}V(y;k)\,\frac{\partial\overline{V(y;k)}}{\partial n(y)}\,\mathrm{d}s(y)\right). \tag{2.26}$$

If we use the boundary condition (2.18) in Eq. (2.26) we get

$$0=\operatorname{Im}\left(\overline{\zeta_e}\int_{\partial D}\zeta_i(y)\,W(y;k)\frac{\partial\overline{W(y;k)}}{\partial n(y)}\,\mathrm{d}s(y)\right). \tag{2.27}$$

Multiplying Eq. (2.25) by $\overline{\zeta_e}$ and taking its imaginary part we get

$$\begin{aligned}&\operatorname{Im}\left(\overline{\zeta_e}\int_{\partial D}\zeta_i(y)\,W(y;k)\frac{\partial\overline{W(y;k)}}{\partial n(y)}\,\mathrm{d}s(y)\right)\\&=\operatorname{lm}\left(\overline{\zeta_e}\int_D\zeta_i(y)\,|\nabla W(y;k)|^2\,\mathrm{d}y\right)+\operatorname{Im}\left(\overline{\zeta_e}\int_D W(y)\nabla\zeta_i(y)\,\nabla\overline{W(y)}\,\mathrm{d}y\right)\\&\quad-\operatorname{Im}\left(\overline{\kappa^2\zeta_e}\int_D n(y)\zeta_i(y)\,|W(y;k)|^2\,\mathrm{d}y\right).\end{aligned} \tag{2.28}$$

From Eq. (2.26) the right side of Eq. (2.28) vanishes. Then Eqs. (2.22) and (2.23) imply that W is a constant and from Eq. (2.17) this constant must be zero. Hence $W = V = 0$ and k (or ω) are not transmission eigenvalues.

Note that for the case of physical interest we have $\zeta_i(x) = (a - \mathrm{i}\omega)\rho_i(x)$, $\zeta_e = -\mathrm{i}\omega\rho_e$ and so $\operatorname{Im} \overline{\zeta_e}\zeta_i(x) = \omega a \rho_e \rho_i(x)$,

$$\operatorname{Im} \overline{\zeta_e \kappa^2}\zeta_i(x) = \operatorname{Im}\left[\frac{1}{c_o^2}\rho_e\rho_i(x)\,(a\omega^2 + \omega^4)\right] = 0.$$

Thus, if $a > 0$ and W satisfies Eq. (2.22) then $k > 0$ (or $\omega > 0$) is not a transmission eigenvalue, that is, if $a > 0$ and Eq. (2.23) is valid for $W \in C^2(D) \cap C^1(\overline{D})$, W is a solution of Eq. (2.17). Then $L^2(\partial\Omega) = \overline{\operatorname{span}\,\Im}$.

Lemma 2.2. *Let $F(\hat{x}; k, \alpha)$ be the far-field pattern corresponding to the incident plane wave $u^i(x) = \exp(\mathrm{i}kx\,\alpha)$.*

Then $F(\hat{x}; k, \alpha) = F(-\alpha; k, -\hat{x})$.

Proof. Using the representation theorem for the scattered field u^s and the asymptotic behaviour of the fundamental solution $\Phi(x, y)$, we have

$$F(\hat{x}; k, \alpha) = \frac{1}{4\pi}\int_{\partial D}\left[u_e(y; k, \alpha)\frac{\partial}{\partial n(y)}\mathrm{e}^{-\mathrm{i}k\hat{x}y} - \mathrm{e}^{-\mathrm{i}k\hat{x}y}\frac{\partial u_e(y; k, \alpha)}{\partial n(y)}\right]\mathrm{d}s(y),$$

where $u_e(x; k, \alpha) = u^i(x; k, \alpha) + u^s(x; k, \alpha)$; therefore

$$\begin{aligned} F(\hat{x}; k, \alpha) = {} & \frac{1}{4\pi}\int_{\partial D}\left[\mathrm{e}^{\mathrm{i}ky\alpha}\frac{\partial}{\partial n(y)} - \mathrm{e}^{-\mathrm{i}k\hat{x}y} - \mathrm{e}^{-\mathrm{i}k\hat{x}y}\frac{\partial}{\partial n(y)}\mathrm{e}^{\mathrm{i}ky\alpha}\right]\mathrm{d}s(y) \\ & + \frac{1}{4\pi}\int_{\partial D}\left[u^s(y; k, \alpha)\frac{\partial}{\partial n(y)}\mathrm{e}^{-\mathrm{i}k\hat{x}y} - \mathrm{e}^{-\mathrm{i}k\hat{x}y}\frac{\partial u^s(y; k, \alpha)}{\partial n(y)}\right]\mathrm{d}s(y). \end{aligned} \tag{2.29}$$

From Green's theorem the first integral of the right-hand side of Eq. (2.28) is zero. Then

$$\begin{aligned} F(\hat{x}; k, \alpha) &= \frac{1}{4\pi}\int_{\partial D}\left[u^s(y; k, \alpha)\frac{\partial}{\partial n(y)}\mathrm{e}^{-\mathrm{i}k\hat{x}y} - \mathrm{e}^{-\mathrm{i}k\hat{x}y}\frac{\partial u^s(y; k, \alpha)}{\partial n(y)}\right]\mathrm{d}s(y) \\ &= \frac{1}{4\pi}\int_{\partial D}\left[u^s(y; k, \alpha)\frac{\partial u^i(y; k, -\hat{x})}{\partial n(y)} - u^i(y; k, -\hat{x})\frac{\partial u^s(y; k, \alpha)}{\partial n(y)}\right]\mathrm{d}s(y), \end{aligned} \tag{2.30}$$

where $u^i(y;k,-\hat{x}) = u_e(y;k,-\hat{x}) - u^s(y;k,-\hat{x})$. Now Eq. (2.30) takes the form

$$\begin{aligned} F(\hat{x};k,\alpha) &= \frac{1}{4\pi}\int_{\partial D}\left[u^s(y;k,\alpha)\frac{\partial u_e(y;k,-\hat{x})}{\partial n(y)} - u_e(y;k,-\hat{x})\frac{\partial u^s(y;k,\alpha)}{\partial n(y)}\right] ds(y) \\ &\quad - \frac{1}{4\pi}\int_{\partial D}\left[u^s(y;k,\alpha)\frac{\partial u^s(y;k,-\hat{x})}{\partial n(y)} - u^s(y;k,-\hat{x})\frac{\partial u^s(y;k,\alpha)}{\partial n(y)}\right] ds(y). \end{aligned} \tag{2.31}$$

From Green's theorem the second integral of the right-hand side of Eq. (2.31) is zero. Then

$$F(\hat{x};k,\alpha) = \frac{1}{4\pi}\int_{\partial D}\left[u^s(y;k,\alpha)\frac{\partial u_e(y;k,-\hat{x})}{\partial n(y)} - u_e(y;k,-\hat{x})\frac{\partial u^s(y;k,\alpha)}{\partial n(y)}\right] ds(y),$$

where $u^s(y;k,\alpha) = u_e(y;k,\alpha) - u^i(y;k,\alpha)$. If we use this fact in the last equation then $F(\hat{x};k,\alpha)$ takes the form

$$\begin{aligned} F(\hat{x};k,\alpha) &= \frac{1}{4\pi}\int_{\partial D}\left[u_e(y;k,\alpha)\frac{\partial u_e(y;k,-\hat{x})}{\partial n(y)} - u_e(y;k,-\hat{x})\frac{\partial u_e(y;k,\alpha)}{\partial n(y)}\right] ds(y) \\ &\quad - \frac{1}{4\pi}\int_{\partial D}\left[u^i(y;k,\alpha)\frac{\partial u_e(y;k,-\hat{x})}{\partial n(y)} - u_e(y;k,-\hat{x})\frac{\partial u^i(y;k,\alpha)}{\partial n(y)}\right] ds(y). \end{aligned} \tag{2.32}$$

From Green's theorem the first integral of the right-hand side of the Eq. (2.32) is zero. Then Eq. (2.32) takes the form

$$\begin{aligned} F(\hat{x};k,\alpha) &= -\frac{1}{4\pi}\int_{\partial D}\left[u^i(y;k,\alpha)\frac{\partial u_e(y;k,-\hat{x})}{\partial n(y)} - u_e(y;k,-\hat{x})\frac{\partial u^i(y;k,\alpha)}{\partial n(y)}\right] ds(y) \\ &= -\frac{1}{4\pi}\int_{\partial D}\left[e^{iky\cdot\alpha}\frac{\partial u_e(y;k,-\hat{x})}{\partial n(y)} - u_e(y;k,-\hat{x})\frac{\partial}{\partial n(y)}e^{iky\cdot\alpha}\right] ds(y) \\ &= \frac{1}{4\pi}\int_{\partial D}\left[u_e(y;k,-\hat{x})\frac{\partial}{\partial n(y)}e^{-iky\cdot(-\alpha)} - e^{-iky\cdot(-\alpha)}\frac{\partial u_e(y;k,-\hat{x})}{\partial n(y)}\right] ds(y) \\ &= F(-\alpha;k,-\hat{x}). \end{aligned}$$

Now we consider as an incident wave W^i of the form

$$W^{\mathrm{i}}(x) = \int_{\partial\Omega} \mathrm{e}^{\mathrm{i}kx\cdot\alpha} g(\alpha)\, \mathrm{d}s(\alpha), \quad x \in \mathbb{R}^3, \tag{2.33}$$

where $g \in L^2(\partial\Omega)$ i.e., the incident wave is a Herglotz wave function which is obtained by multiplying $u^i = u_{\mathrm{e}} - u^{\mathrm{s}}$ by $g \in L^2(\partial\Omega)$ and integrating over $\partial\Omega$. So we have

$$\begin{aligned} W^{\mathrm{i}}(x) &= \int_{\partial\Omega} u_i(x;k,\alpha) g(\alpha)\, \mathrm{d}s(\alpha) \\ &= \int_{\partial\Omega} u_{\mathrm{e}}(x;k,\alpha) g(\alpha)\, \mathrm{d}s(\alpha) - \int_{\partial\Omega} u^{\mathrm{s}}(x;k,\alpha) g(\alpha)\, \mathrm{d}s(\alpha). \end{aligned}$$

Suppose that

$$\begin{aligned} W(x) &= \int_{\partial\Omega} u_{\mathrm{e}}(x;k,\alpha) g(\alpha)\, \mathrm{d}s(\alpha). \\ W^{\mathrm{s}}(x) &= \int_{\partial\Omega} u^{\mathrm{s}}(x;k,\alpha) g(\alpha)\, \mathrm{d}s(\alpha). \end{aligned} \tag{2.34}$$

Then W satisfies

$$\begin{aligned} &\Delta W + k^2 W = 0 \text{ in } x \in \mathbb{R}^3 \backslash \overline{D}, \\ &\lim_{r\to\infty} r\left(\frac{\partial W^{\mathrm{s}}}{\partial r} - \mathrm{i}kW^{\mathrm{s}}\right) = 0. \end{aligned} \tag{2.35}$$

If we insert Eq. (2.11) into Eq. (2.34) we obtain

$$W^{\mathrm{s}}(x) = \frac{\mathrm{e}^{\mathrm{i}kr}}{r} \int_{\partial\Omega} F(\hat{x};k,\alpha) g(\alpha)\, \mathrm{d}s(\alpha) + \mathrm{o}\left(\frac{1}{r^2}\right). \tag{2.36}$$

Therefore W^{s} has the far-field pattern for W^{i}

$$\int_{\partial\Omega} F(\hat{x};k,\alpha) g(\alpha)\, \mathrm{d}s(\alpha) \tag{2.37}$$

Lemma 2.3. *The orthogonal complement of* $\mathfrak{I}$ in $L^2(\partial\Omega)$ *consists of those functions* $g \in L^2(\partial\Omega)$ *for which there exists* $W \in C^2(D) \cap C^1(\mathbb{R}^3\backslash\overline{D})$, *and* V *defined by Eq.* (2.13) *such that*

$$\begin{aligned} &\Delta W + k^2 W = 0 \text{ in } \mathbb{R}^3 \backslash \overline{D}, \\ &\Delta V + k^2 V = 0 \text{ in } D, \end{aligned} \tag{2.38}$$

$$\begin{aligned} &V = W \text{ on } \partial D, \\ &\frac{\partial V}{\partial n} = \frac{\partial W}{\partial n} \text{ on } \partial D. \end{aligned} \tag{2.39}$$

Proof. Suppose that $h \in \mathfrak{I}^{\perp}$, i.e.,

$$\int_{\partial\Omega} F(\hat{x}; k, \alpha)\overline{h(\hat{x})}\mathrm{d}s(\hat{x}) = 0. \tag{2.40}$$

From Lemma 2.2 we have that

$$\int_{\partial\Omega} F(-\alpha; k, \hat{x})\overline{h(\hat{x})}\mathrm{d}s(\hat{x}) = 0 \tag{2.41}$$

for all $\alpha \in \partial\Omega$ i.e.,

$$\int_{\partial\Omega} F(\hat{x}; k, \alpha)\overline{h(-\alpha)}\mathrm{d}s(\alpha) = 0 \tag{2.42}$$

for all $\hat{x} \in \partial\Omega$. Note that here the left-hand side of Eq. (2.42) is the far-field pattern of the scattered field W^{s} corresponding to the incident field W^{i} for $g(\alpha) = \overline{h(-\alpha)} \in L^2(\partial\Omega)$. But from Eq. (2.42) we have that the far-field pattern of W^{s} vanishes and from [4], corollary 3.9, W^{s} vanishes in $\mathbb{R}^3 \backslash \overline{D}$. Then in ∂D $W = W^{\mathrm{i}}$ and $\partial W/\partial n = \partial W^{\mathrm{i}}/\partial n$. This completes the proof of the Lemma.

Lemma 2.4. *Suppose that for all x in D the inequality*

$$\mathrm{Im}\left(\kappa^2 W \nabla\rho_i(x) \cdot \nabla\overline{W(x)}\right) > 0 \tag{2.43}$$

is fulfilled. Then k (or ω) is not a transmission eigenvalue for the homogeneous interior transmission problem.

Proof. If k is a transmission eigenvalue for the homogeneous interior transmission problem, then there exists a Herglotz wave function V and a function $W \in C^2(D) \cap C^1(\overline{D})$ such that

$$\Delta V + k^2 V = 0 \text{ in } D$$
$$\Delta W + \kappa^2 n(x) W = 0 \text{ in } D \tag{2.44}$$

$$\zeta_i(x) W - \zeta_e V = 0 \text{ on } \partial D$$
$$\frac{\partial}{\partial n}(W - V) = 0 \text{ on } \partial D. \tag{2.45}$$

From Green's second identity

$$0 = \int_D \left(V \nabla^2 \bar{V} - \bar{V} \nabla^2 V \right) \mathrm{d}y = \int_{\partial D} \left(V \frac{\partial \bar{V}}{\partial n} - \bar{V} \frac{\partial \bar{V}}{\partial n} \right) \mathrm{d}s(y)$$
$$= \frac{1}{\rho_e} \int_{\partial D} \rho_i(y) 2 \operatorname{Im} \left\{ \left(1 + \mathrm{i} \frac{a}{\omega} \right) W(y) \frac{\partial \overline{W(y)}}{\partial n} \right\} \mathrm{d}s(y)$$

or

$$0 = \operatorname{Im} \left\{ \left(1 + \mathrm{i} \frac{a}{\omega} \right) \int_{\partial D} \rho_i(y) W(y) \frac{\partial \overline{W(y)}}{\partial n} \, \mathrm{d}s(y) \right\}. \tag{2.46}$$

Now if we use Green's first identity in Eq. (2.46) we get

$$0 = \operatorname{Im} \int_D \left(1 + \mathrm{i} \frac{a}{\omega} \right) \left(\overline{-\kappa^2 n(y)} \right) \rho_i(y) W(y) \mathrm{d}y$$
$$+ \operatorname{Im} \int_D \left(1 + \mathrm{i} \frac{a}{\omega} \right) \rho_i(y) |\nabla W(y)|^2 \, \mathrm{d}y$$
$$+ \operatorname{Im} \int_D \left(1 + \mathrm{i} \frac{a}{\omega} \right) W(y) \nabla \rho_i(y) \cdot \nabla \overline{W(y)} \, \mathrm{d}y. \tag{2.47}$$

Note that for the case of physical interest we have

$$\operatorname{Im} \left(1 + \mathrm{i} \frac{a}{\omega} \right) \overline{(\kappa^2 n(x))} = \operatorname{Im} \left[-\frac{1}{c(x)^2} (\omega^2 + a^2) \right] = 0$$

$$\operatorname{Im} \int_D \left(1 + \mathrm{i} \frac{a}{\omega} \right) \rho_i(y) |\nabla W(y)|^2 \mathrm{d}y = \frac{a}{\omega} \int_D \rho_i(y) |\nabla W(y)|^2 \, \mathrm{d}y$$

and

$$\operatorname{Im}\int_D \left(1+\mathrm{i}\frac{a}{\omega}\right) W(y)\nabla\rho_i(y)\cdot\nabla\overline{W(y)}\,\mathrm{d}y = \frac{1}{k^2}\operatorname{Im}\int_D \kappa^2 W(y)\nabla\rho_i(y)\cdot\nabla\overline{W(y)}\,\mathrm{d}y.$$

So that in D, V and W must be equal to zero and the lemma is proven.

3. Interior transmission problem

Now we consider the

$$\begin{aligned}&\Delta V + k^2 V = 0 \text{ in } D,\\ &\Delta W + k^2 n(x) W = 0 \text{ in } D,\end{aligned} \tag{3.1}$$

$$\begin{aligned}&\rho_e V - \rho_i(x) W = \rho_e \frac{\mathrm{e}^{\mathrm{i}kr}}{r} \text{ on } \partial D,\\ &\frac{\partial}{\partial n}(V-W) = \frac{\partial}{\partial n}\frac{\mathrm{e}^{\mathrm{i}kr}}{r} \text{ on } \partial D.\end{aligned} \tag{3.2}$$

Inhomogeneous interior transmission problem for V, $W \in C^2(D)\cap C^1(\overline{D})$.

If $\{V, W\}$ is a solution of the interior transmission problem (3.1), (3.2) and V is a Herglotz wave function with Herglotz kernel $g \in L^2(\partial\Omega)$, then with Eq. (2.14)

$$\int_{\partial\Omega} F(\hat{x};k,\alpha)\overline{g(\hat{x})}\,\mathrm{d}s(\hat{x}) = \frac{1}{4\pi\rho_e}\int_{\partial D}\rho_i(y)\left[u_i(y)\frac{\partial\overline{W(y)}}{\partial n} - \overline{W(y)}\frac{\partial u_i(y)}{\partial n}\right]\mathrm{d}s(y) + 1. \tag{3.3}$$

Note that from Eq. (3.3) we see that if V, W is a solution of Eqs. (3.1) and (3.2) and V can be approximated arbitrarily closely in $C^1(\overline{D})$ by a Herglotz wave function, then Eq. (3.3) is approximately true in the sense that for every $\varepsilon > 0$ there exists a Herglotz kernel $g_\varepsilon \in L^2(\partial\Omega)$ such that

$$\left|\int_{\partial\Omega} F(\hat{x};k,\alpha)\overline{g_\varepsilon(\hat{x})}\,\mathrm{d}s(\hat{x}) - \frac{1}{4\pi\rho_e}\int_{\partial D}\rho_i(y)\left[u_i(y)\frac{\partial\overline{W(y)}}{\partial n} - \overline{W(y)}\frac{\partial u_i(y)}{\partial n}\right]\mathrm{d}s(y) - 1\right| < \varepsilon.$$

The existence of a unique solution V, W to Eqs. (3.1) and (3.2) is addressed by the following theorem.

Theorem 3.1. *Assume that $\rho_e \neq \rho_i(x)$ for $x \in \overline{D}$ and k (or ω) is not a transmission eigenvalue. Then there exists unique solution V, W of the interior transmission problem.*

Proof. We can rewrite Eqs. (3.1) and (3.2) as the system

$$\begin{aligned}&(\nabla^2 - I)(V - W) = k^2[n(x)W - V] + W - V \quad \text{in } D,\\ &\frac{\partial}{\partial n}(V - W) = \frac{\partial}{\partial n}\frac{e^{ikr}}{r} \quad \text{on } \partial D\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}&(\nabla^2 - I)(\rho_e V - \rho_i(x)W) = k^2[n(x)\rho_i(x)W - \rho_e V]\\ &- 2\nabla\rho_i(x)\cdot\nabla W(x) - W\nabla^2\rho_i(x) - \rho_e V + \rho_i(x)W \quad \text{in } D\\ &\rho_e V - \rho_i(x)W = \rho_e\frac{e^{ikr}}{r} \quad \text{on } \partial D.\end{aligned} \tag{3.5}$$

Let $N(x,y)$ be the Neumann function for the operator $\nabla^2 - I$ in D [8]. Consider the problem Eq. (3.4), we have

$$\begin{aligned}V(x) - W(x) = &-\int_D \frac{\partial}{\partial n}\frac{e^{ikr}}{r}N(x,y)\,ds(y) + \int_{\partial D} N(x,y)\\ &\times\{k^2[n(y)W - V] + W - V\}\,dy.\end{aligned} \tag{3.6}$$

Let $G(x,y)$ be the Green's function for the operator $\nabla^2 - I$ in D [8]. Consider the problem (3.5), we have

$$\begin{aligned}\rho_e V(x) - \rho_i(x)W(x) = &\int_{\partial D}\rho_e\frac{e^{ikr}}{r}\frac{\partial G(x,y)}{\partial n(y)}\,ds(y)\\ &+\int_D G(x,y)\{k^2[n(y)\rho_i(y)W - \rho_e V] - 2\nabla\rho_i(y)\nabla W - W\nabla^2\rho_i(y)\\ &-\rho_e V + \rho_i(y)W\}\,dy.\end{aligned} \tag{3.7}$$

Hence the problem (3.4) and (3.5) is reduced to the integral equations (3.6) and (3.7) respectively.

Let

$$\phi_1(x) = -\int_{\partial D}\frac{\partial}{\partial n(y)}\frac{e^{ikr}}{r}N(x,y)\,ds(y)$$

$$\phi_2(x) = \int_{\partial D}\rho_e\frac{e^{ikr}}{r}\frac{\partial G(x,y)}{\partial n(y)}\,ds(y)$$

Now introduce the integral operators defined by $K_i: C^1(\overline{D}) \to C(\overline{D})$, $i = 1,2,3,4$.

$$(K_1V)(x) = -(k^2+1)\int_D N(x,y)V(y)\,\mathrm{d}y$$

$$(K_2W)(x) = \int_D N(x,y)[k^2n(y)+1]W(y)\,\mathrm{d}y$$

$$(K_3V)(x) = -\rho_e(k^2+1)\int_D G(x,y)V(y)\,\mathrm{d}y$$

$$(K_4W)(x) = \int_D G(x,y)\{[k^2n(y)\rho_i(y) - \nabla^2\rho_i(y) + \rho_i(y)]W(y) - 2\nabla\rho_i(y)\nabla W(y)\}\,\mathrm{d}y.$$

The operators K_i are compact operators [10] having the Green or Neumann function as kernel and ϕ_1, ϕ_2 are given functions in D which depend only on the boundary data. We can rewrite Eqs. (3.6) and (3.7) as matrix equation form.

$$A\begin{pmatrix} V \\ W \end{pmatrix} = K\begin{pmatrix} V \\ W \end{pmatrix} + \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},$$

where

$$A = \begin{pmatrix} 1 & -1 \\ \rho_e & \rho_i(x) \end{pmatrix}, \quad K_i = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}.$$

Since $\rho_e = \rho_i(x)$ then A is a regular matrix. The homogeneous problem (3.8) with $\phi_1 = \phi_2 = 0$ is equivalent to the transmission problem (2.16), (2.17) with $\kappa = k$, $\rho_e = \zeta_e, \rho_i(x) = \zeta_i(x)$. Hence if k (or ω) is not a transmission eigenvalue, then by Theorem 2.1 the integral equation (3.8) has a unique solution.

Example. Let D be unit disc and $\kappa^2 n(x) = \kappa^2 - P(x)$, where

$$P(x) = -\frac{f''(r)}{f(r)} + 2\frac{f'(r)^2}{f(r)^2} - 2\kappa\,\cot\kappa r\frac{f'(r)}{f(r)}.$$

Let a_o be a non-zero constant. Then the function $W(x) = a_o(\sin\kappa r/rf(r))$ satisfies $\Delta W + \kappa^2 n(x)\ W = 0$ for $x \in D$. The function $V(x) = b_o(\sin kr/r)$ satisfies $\Delta V + k^2 V = 0$ in $\mathbb{R}^3\backslash\overline{D}$, where b_o is a non-zero constant.

Let $\rho_i(x) = \rho(|x|) = \rho_i(r)$ and

$$M = \det \begin{pmatrix} -\rho_i(1)\dfrac{\sin\kappa}{f(1)} & \rho_e \sin\kappa \\ \kappa\dfrac{\cos\kappa}{f(1)^2} - \dfrac{\sin\kappa}{f(1)} + \dfrac{f'(1)\sin\kappa}{f(1)^2} & k\ \cos k - \sin k \end{pmatrix}.$$

If $M=0$ and a_o, b_o is a solution of

$$\begin{aligned} &-a_o\rho_i(1)\frac{\sin\kappa}{f(1)} + b_o\rho_e\ \sin\kappa = 0 \\ &a_o\left(\kappa\frac{\cos\kappa}{f(1)^2} - \frac{\sin\kappa}{f(1)} + \frac{f'(1)\sin\kappa}{f(1)^2}\right) + b_o(k\ \cos k - \sin k) = 0. \end{aligned} \tag{3.8a}$$

Then V and W satisfy

$$\begin{aligned} &\Delta V + k^2 V = 0 \text{ in } \mathbb{R}^3\backslash\overline{D} \\ &\Delta W + \kappa^2 n(x) W = 0 \text{ in } D \\ &\rho_e V - \rho_i(r) W = 0 \text{ on } \partial D \\ &\rho_e V - \rho_i(r) W = 0 \text{ on } \partial D \\ &\frac{\partial V}{\partial n} - \frac{\partial W}{\partial n} = 0 \text{ on } \partial D \end{aligned}$$

the homogeneous transmission problem.

If $M \neq 0$ and a_o, b_o is a solution of

$$\begin{aligned} &-a_o\rho_i(1)\frac{\sin\kappa}{f(1)} + b_o\rho_e\ \sin\ \kappa = \rho_e e^{ik} \\ &a_o\left(\kappa\frac{\cos\kappa}{f(1)^2} - \frac{\sin\kappa}{f(1)} + \frac{f'(1)\sin\kappa}{f(1)^2}\right) + b_o(k\ \cos k - \sin k) = (ik-1)e^{ik}. \end{aligned} \tag{3.8b}$$

Then V and W satisfy1

$$\begin{aligned} &\Delta V + k^2 V = 0 \text{ in } \mathbb{R}^3\backslash\overline{D} \\ &\Delta W + \kappa^2 n(x) W = 0 \text{ in } D \\ &\rho_e V - \rho_i(r) W = \rho_e \frac{e^{ikr}}{r} \text{ on } \partial D \\ &\frac{\partial V}{\partial n} - \frac{\partial W}{\partial n} = \frac{\partial}{\partial n}\frac{e^{ikr}}{r} \text{ on } \partial D \end{aligned}$$

the inhomogeneous transmission problem.

4. Integral equation method

Let W be a solution of Eqs. (3.1) and (3.2). Then from Green's formula we have for $x \in D$

$$\begin{aligned} W(x) &= \int_{\partial D} \left[\Phi(x,y) \frac{\partial W(y)}{\partial n} - W(y) \frac{\partial \Phi(x,y)}{\partial n(y)} \right] \mathrm{d}s(y) \\ &\quad - k^2 \int_D \Phi(x,y)\, m(y)\, W(y)\, \mathrm{d}y \\ &= W_{\mathrm{o}}(x) - k^2 (TW)(x), \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} m(x) &= 1 - n(x) \\ \Phi(x,y) &= -\frac{1}{4\pi} \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{|x-y|}, \quad x \neq y \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} W_{\mathrm{o}}(x) &= \int_{\partial D} \left[\Phi(x,y) \frac{\partial W(y)}{\partial n} - W(y) \frac{\partial \Phi(x,y)}{\partial n(y)} \right] \mathrm{d}s(y) \\ TW &= \int_D \Phi(x,y) m(y) W(y)\, \mathrm{d}y. \end{aligned} \tag{3.11}$$

Note that W_{o} is a solution of $\Delta W_{\mathrm{o}} + k^2 W_{\mathrm{o}} = 0$ in D and continuously assumes boundary data in class $C^{2,\alpha}(\partial D)$.

Colton and Monk [6] showed that $(I + k^2 T)^{-1}$ exists and from [7] T can be the contraction operator, now we have the following theorem.

Theorem 4.1. *Assume that W is a solution of the problem* (3.1), (3.2) *and $W_{\mathrm{o}}(x)$ is a given function with Eq.* (3.11) *corresponding to the $g_{\mathrm{o}} \in L^2(\partial\Omega)$. Then the integral equation* (3.9) *has a unique solution in $C^1(\overline{D})$.*

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