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ORIGINAL PAPER

Generalizing the Fourier Transform for Weighted Differential Operators

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Abstract
In this study, a weighted generalization of the Fourier transform using a weight function w with respect to a function g is defined and some fundamental properties are examined. Then, the existence of the weighted Fourier transform is proven, and the transforms of some example functions are calculated to facilitate understanding of the transform, and then the inverse of the weighted Fourier transform is given. Furthermore, a new weighted convolution operation is defined, and a weighted convolution theorem is proven. Finally, the weighted Fourier transform of the weighted derivative and the fractional weighted derivative of a function f with a weighted w respect to another function g is investigated. Additionally, the certain theorems are verified with examples, and our study is enriched with numerical graphs.

Keywords Weighted fractional integrals · Weighted fractional derivatitives · Fourier transform · Integral transforms

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1 Introduction

The Fourier transform [1] is one of the fundamental mathematical tools used in engineering, physics, and many other applied sciences. Its application areas also encompass various scientific disciplines such as signal processing, telecommunications, audio and video analysis, and medical imaging. Discrete Fourier transform and Fast Fourier transform [2, 3] are its most important variants, and have optimized computational procedures that enable real-time analysis in many applications. Significant research efforts on the fractional Fourier transform have also emerged in recent times [4–6].

The Fourier transform [1, 7] of an absolutely integrable function f is defined as

$$\mathcal{T}(f) = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-ikt}\, dt, \quad k \in \mathbb{R}. \tag{1.1}$$

Also, the inverse Fourier transform is given by

$$\mathcal{T}^{-1}(F) = f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)\, e^{ikt}\, dk, \quad t \in \mathbb{R}. \tag{1.2}$$

where F is a function of Schwartz class on $\mathbb{R}$ [8]. Recall that a function f is of Schwartz class on $\mathbb{R}$ denoted by $\mathcal{S}(\mathbb{R})$ if and only if $f \in C^{\infty}(\mathbb{R})$ and all derivatives of f are rapidly decreasing, namely,

$$\sup_{t\in\mathbb{R}} |t|^m \left| f^{(n)}(t) \right| < \infty \quad \text{for all } m, n \in \mathbb{N}_0 \tag{1.3}$$

where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ [8, p. 134].

Let us recall that the w-weighted first and $n-$th order derivatives with respect to the function g of a function f are defined as follows

$$\left(\mathfrak{D}_{w,g}\, f\right)(x) = \frac{1}{w(x)} \frac{1}{g'(x)} D_x \Big(w(x) f(x) \Big), \qquad w(x) \neq 0, \tag{1.4}$$

$$\left(\mathfrak{D}^n_{w,g}\, f\right)(x) = \frac{1}{w(x)} \left(\frac{1}{g'(x)} D_x \right)^n \Big(w(x) f(x) \Big), \qquad w(x) \neq 0,\ n \in \mathbb{N}, \tag{1.5}$$

where $D_x = d/dx$ and g represents a differentiable and strictly increasing function [9, 10], respectively.

Fractional calculus has become increasingly important in mathematics and applied sciences in recent years. The Riemann–Liouville and Caputo fractional derivatives are two of the most used operators [11]. In [12], the generalization of the Laplace transform is obtained, and kernels of these fractional operators are dependent on a function on the space of absolute continuous functions. These operators can be regarded as special cases of the generalized fractional derivatives proposed by Agrawal [9], who detailed their properties in his work but did not elaborate on the spaces where these operators are defined. In reference [10], the spaces in which the operators introduced in Agrawal's work are defined have been discussed, and the weighted Laplace transform has been

introduced. Additionally, references [13, 14] have provided weighted generalizations of certain fractional integrals and derivatives.

The fractional integral [10] of a function f with respect to a function g with weight w is defined by

$$\left(\mathfrak{J}^{\alpha}_{w,g}\, f\right)(x) = \frac{1}{\Gamma(\alpha)\, w(x)} \int_0^x \left[g(x) - g(t)\right]^{\alpha-1} w(t)\, f(t)\, g'(t)\, dt, \quad w(x) \neq 0 \tag{1.6}$$

where Γ is Euler's Gamma function and, g is strictly increasing and differentiable. The n-th order w-weighted Caputo fractional derivative [10] of a function f with respect to a function g is defined by

$$\left(\mathfrak{C}^{\alpha}_{w,g}\, f\right)(x) = \frac{1}{\Gamma(n-\alpha)\, w(x)} \int_0^x \left[g(x) - g(t)\right]^{n-\alpha-1} w(t) \left[\mathfrak{D}^{n}_{w,g}\, f(t)\right] g'(t)\, dt, \tag{1.7}$$

where $w(x) \neq 0$ and $n-1 < \alpha \leq n$. The n-th order w-weighted Riemann-Liouville fractional derivative of a function f with respect to g is given by

$$\left(\mathfrak{R}^{\alpha}_{w,g}\, f\right)(x) = \mathfrak{D}^{n}_{w,g} \left(\mathfrak{J}^{n-\alpha}_{w,g}\, f\right)(x), \quad w(x) \neq 0 \tag{1.8}$$

where $n-1 < \alpha \leq n$ [9].

Let us also note that the following equations are valid:

$$\left(\mathfrak{J}^{\alpha}_{w,g}\, f\right)(x) = \frac{1}{\Gamma(\alpha)\, w(x)} \int_{-\infty}^{\infty} u(t)u(x-t) \left[g(x) - g(t)\right]^{\alpha-1} w(t)\, f(t)\, g'(t)\, dt, \tag{1.9}$$

$$\left(\mathfrak{C}^{\alpha}_{w,g}\, f\right)(x) = \frac{1}{\Gamma(n-\alpha)\, w(x)} \int_{-\infty}^{\infty} u(t)u(x-t) \left[g(x) - g(t)\right]^{n-\alpha-1} w(t) \left[\mathfrak{D}^{n}_{w,g}\, f(t)\right] g'(t)\, dt \tag{1.10}$$

where $u(t)$ is the unit step function.

In the upcoming section, we will define an integral transform and discuss its fundamental properties.

2 The weighted Fourier transform

We start with the following definition of a normed vector space to guarantee the existence of the integral transform, we are going to construct. Now, let us extend the weighted function space defined on $[a, b]$ given in [10] to the whole real line.

Definition 2.1 Let $g, w : \mathbb{R} \to \mathbb{R}$ such that the function g is strictly increasing differentiable function on $\mathbb{R}$ and, $w(x) \neq 0$ for all $x \in \mathbb{R}$. A function f belongs to $L^{p}_{w,g}$, namely, the space of $w-$weighted Lebesgue measurable functions with respect to the

function g on $\mathbb{R}$, if and only if $\|f\|_{p,w,g} < \infty$ where

$$\|f\|_{p,w,g} = \left(\int_{-\infty}^{\infty} |w(x) f(x)|^p \, d\mu_g(x) \right)^{\frac{1}{p}}, \qquad for \;\; 1 \leq p < \infty \tag{2.1}$$

and

$$\|f\|_{\infty,w,g} = ess\ sup_{\mu_g} \{|w(x) f(x)| : x \in \mathbb{R}\}. \tag{2.2}$$

Here, the measure is induced by g, namely $d\mu_g(x) = g'(x)dx$.

Now, we present the definition of the integral transform we are working on, which forms the most crucial cornerstone of this paper.

Definition 2.2 Let $g, w : \mathbb{R} \to \mathbb{R}$ and the function g is strictly increasing differentiable function on $\mathbb{R}$ satisfying $g(x) \to \pm\infty$ when $x \to \pm\infty$. So, we introduce the weighted Fourier transform of a function f belonging to the class $L^1_{w,g}$ as follows:

$$\mathcal{F}_g^w\{f\} = \int_{-\infty}^{\infty} w(t) f(t) \, e^{-ikg(t)} \, g'(t) \, dt = \psi(k), \quad k \in \mathbb{R}. \tag{2.3}$$

Remark 2.3 Let $f, h \in L^1_{w,g}$. Then

$$\mathcal{F}_g^w\{\alpha f \ + \beta h\} = \alpha \, \mathcal{F}_g^w\{f\} + \beta \, \mathcal{F}_g^w\{h\} \tag{2.4}$$

where $\alpha, \beta \in \mathbb{R}$. In the other words, the weighted Fourier transform is linear.

Remark 2.4 If we take $w(x) = 1/\sqrt{2\pi}$ and $g(x) = x$ in the definition of the weighted Fourier transform given in (2.3), clearly weighted transform will turn into the well-known Fourier transform given by $\mathcal{T}$ in the equality (1.1). In addition, the following equalities are achieved

$$\mathcal{F}_g^w\{f\} = \sqrt{2\pi} \, \mathcal{T}\left\{(wf) \circ g^{-1}\right\} \quad \text{and} \quad \mathcal{F}_g^w\left\{\frac{f \circ g}{w}\right\} = \sqrt{2\pi} \, \mathcal{T}\{f\}. \tag{2.5}$$

Remark 2.5 If the weight function is chosen as $w(x) = 1$ in the definition of the weighted Fourier transform given in (2.3), it reduces to the Fourier transform with respect to another function described in [15] given by

$$\mathcal{F}_g\{f\} = \int_{-\infty}^{\infty} f(t) \, e^{-ikg(t)} \, g'(t) \, dt. \tag{2.6}$$

Note that the fractional version of the transform defined in [15] has also been studied in [16].

Theorem 2.6 *If $f \in L^1_{w,g}$, then the weighted Fourier transform of the function f exists.*

Proof Using the triangle inequality and the fact that $\left|e^{-ikg(t)}\right| = 1$, we easily obtain

$$\begin{aligned}\left|\mathcal{F}_g^w\{f\}\right| &\leq \left|\int_{-\infty}^{\infty} w(t)\, f(t)\, e^{-ikg(t)}\, g'(t)\, dt\right| \\ &\leq \int_{-\infty}^{\infty} \left|w(t)\, f(t)\, e^{-ikg(t)}\, g'(t)\right| dt \\ &\leq \int_{-\infty}^{\infty} |w(t)\, f(t)|\, g'(t) dt \\ &\leq \|f\|_{1,w,g} \\ &< \infty.\end{aligned} \tag{2.7}$$

Thus, the existence of the Fourier transform of the function f is established. □

In order to gain a deeper understanding of the definition of the weighted Fourier transform, let's review the two examples presented below:

Example 2.7 Let

$$f(x) = a\left(x^6 + 2x^4 + x^2\right) + b\left(x^3 + x\right) + c, \quad g(x) = x^3 + x \quad w(x) = e^{-(x^6+2x^4+x^2)} \tag{2.8}$$

where $a, b, c \in \mathbb{R}$. Now, let's compute the weighted Fourier transform of the function f. We have

$$\begin{aligned}\mathcal{F}_g^w\{f\} &= \int_{-\infty}^{\infty} w(x)\, f(x)\, e^{-ikg(x)}\, g'(x)\, dx \\ &= \int_{-\infty}^{\infty} e^{-(x^6+2x^4+x^2)} \left[a\left(x^6 + 2x^4 + x^2\right) + b\left(x^3 + x\right) + c\right] e^{-ik(x^3+x)}\, (3x^2 + 1)\, dx \\ &= \int_{-\infty}^{\infty} e^{-t^2}\left[at^2 + bt + c\right] e^{-ikt}\, dt\end{aligned} \tag{2.9}$$

using the substitution $t = x^3 + x$. According to [7, p. 21], the integral corresponding to the Fourier transform of $e^{-t^2}\sqrt{2\pi}$ yields

$$\int_{-\infty}^{\infty} e^{-t^2} e^{-ikt}\, dt = \sqrt{\pi} e^{-k^2/4}. \tag{2.10}$$

By differentiating both sides of (2.10) with respect to k once and twice, we obtain

$$\int_{-\infty}^{\infty} t e^{-t^2} e^{-ikt}\, dt = -\frac{ik\sqrt{\pi}}{2} e^{-k^2/4}, \tag{2.11}$$

$$\int_{-\infty}^{\infty} t^2 e^{-t^2} e^{-ikt}\, dt = \sqrt{\pi}\left(\frac{1}{2} - \frac{k^2}{4}\right) e^{-k^2/4}. \tag{2.12}$$

Substituting the values from (2.10), (2.11), and (2.12) into (2.9), we deduce that

$$\mathcal{F}_g^w\{f\} = \frac{\sqrt{\pi}}{4}\, e^{-k^2/4}\left[a(2-k^2) - 2bik + 4c\right]. \tag{2.13}$$

Example 2.8 Let us assume

$$f(x) = 1, \quad g(x) = x, \quad w(x) = e^{-a|x|} \tag{2.14}$$

where $a > 0$. Our next step is to compute the weighted Fourier transform of f. It is

$$\begin{aligned}\mathcal{F}_g^w\{f\} &= \int_{-\infty}^{\infty} w(x)\, f(x)\, e^{-ikg(x)}\, g'(x)\, dx \\ &= \int_0^{\infty} e^{-(ik+a)x}\, dx + \int_{-\infty}^{0} e^{(a-ik)x}\, dx \\ &= \frac{2a}{a^2+k^2}.\end{aligned}$$

Now, we formulate the inverse of the weighted Fourier transform through the following theorem.

Theorem 2.9 *If the inverse of the weighted Fourier transform of a function f exists, it is defined as:*

$$\left(\mathcal{F}_g^w\right)^{-1}\{f\} = \frac{1}{2\pi\, w(t)} \int_{-\infty}^{\infty} f(k)\, e^{ikg(t)}\, dk = \phi(t), \quad t \in \mathbb{R}. \tag{2.15}$$

Proof Let us assume that $\mathcal{F}_g^w\{f\} = F(k)$. By using Remark 2.4 and the equality in (1.2), we have

$$\begin{aligned}\left((wf)\circ g^{-1}\right)(x) &= \frac{1}{\sqrt{2\pi}}\mathcal{T}^{-1}\{F(k)\}(x) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk.\end{aligned} \tag{2.16}$$

When $x = g(t)$ is substituted in (2.16), we derive

$$f(t) = \frac{1}{2\pi\, w(t)} \int_{-\infty}^{\infty} F(k)\, e^{ikg(t)}\, dk. \tag{2.17}$$

Consequently, the intended outcome is achieved. □

Lemma 2.10 [8, p. 137] *If $f \in \mathcal{S}(\mathbb{R})$, then $\mathcal{T}^{-1}(f) \in \mathcal{S}(\mathbb{R})$.*

If we consider Lemma 2.10 and Remark 2.4 together, we can clearly state the following theorem without proof. This theorem also shows that the weighted Fourier transform has an inverse.

Theorem 2.11 *If* $(wf)\circ g^{-1}\in\mathcal{S}(\mathbb{R})$, *then* $\left(\mathcal{F}_g^w\right)^{-1}\{f\}\in\mathcal{S}(\mathbb{R})$.

When considering the product of two functions, their weighted Fourier transform is not equivalent to the product of their individual weighted Fourier transforms. To address this, we define a convolution product as follows:

Definition 2.12 The weighted convolution of functions f and h is defined as follows:

$$\left(f\ *_g^w\ h\right)(x)=\frac{1}{w(x)}\int_{-\infty}^{\infty}w\Big(g^{-1}\big(g(x)-g(t)\big)\Big)f\Big(g^{-1}\big(g(x)-g(t)\big)\Big)w(t)h(t)\,g'(t)\,dt \tag{2.18}$$

where $f,h\in L^1_{w,g}$ and the function g is strictly increasing differentiable function on $\mathbb{R}$ satisfying $g(x)\to\pm\infty$ when $x\to\pm\infty$.

The functions f, h and y provide the following explicit properties of weighted convolution:

i) Commutativity: $f\ *_g^w\ h=h\ *_g^w\ f$,
ii) Associativity: $f\ *_g^w\ (h\ *_g^w\ y)=(f\ *_g^w\ h)\ *_g^w\ y$,
iii) Distributivity: $f\ *_g^w\ (h+y)=(f\ *_g^w\ h)\ +\ (f\ *_g^w\ y)$,
iv) Scalar Multiplication: $\alpha\ (f*_g^w\ h)=(\alpha f)\ *_g^w\ h=f\ *_g^w\ (\alpha h),\quad \alpha\in\mathbb{R}$.

Theorem 2.13 *There exists*

$$\mathcal{F}_g^w\{f\ *_g^w\ h\}=\mathcal{F}_g^w\{f\}\mathcal{F}_g^w\{h\} \tag{2.19}$$

where $f,h\in L^1_{w,g}$.

Proof We have

$$\begin{aligned}\mathcal{F}_g^w\{f\ *_g^w\ h\}&=\int_{-\infty}^{\infty}\frac{1}{w(x)}\Bigg(\int_{-\infty}^{\infty}w\Big(g^{-1}\big(g(x)-g(t)\big)\Big)f\Big(g^{-1}\big(g(x)-g(t)\big)\Big)\\&\qquad\times\ w(t)h(t)\,g'(t)\,dt\Bigg)e^{-ikg(x)}\,w(x)g'(x)\,dx\\&=\int_{-\infty}^{\infty}\Bigg(\int_{-\infty}^{\infty}w\Big(g^{-1}\big(g(x)-g(t)\big)\Big)f\Big(g^{-1}\big(g(x)-g(t)\big)\Big)\\&\qquad\times\ e^{-ikg(x)}\,g'(x)\,dx\Bigg)w(t)h(t)\,g'(t)\,dt\end{aligned} \tag{2.20}$$

if we use the definition of weighted convolution and change the order of integration in the above integral. Then, if we change the variable $z=g^{-1}\,(g(x)-g(t))$ in the first integration above in (2.20), we have that

$$\begin{aligned}\mathcal{F}_g^w\{f \ *_g^w \ h\} &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} w(z)f(z)\,e^{-ik[g(z)+g(t)]}\,g'(z)dz\right)w(t)h(t)\,g'(t)\,dt\\ &= \left(\int_{-\infty}^{\infty} w(z)f(z)\,e^{-ikg(z)}\,g'(z)\,dz\right)\left(\int_{-\infty}^{\infty} w(t)h(t)\,e^{-ikg(t)}\,g'(t)\,dt\right)\\ &= \mathcal{F}_g^w\{f\}\mathcal{F}_g^w\{h\}. \end{aligned} \tag{2.21}$$

□

Theorem 2.14 *Under the condition that* $\lim_{x\to\pm\infty} w(x)f(x) = 0$, *we have*

$$\mathcal{F}_g^w\{\mathfrak{D}_{w,g}\,f\} = ik\,\mathcal{F}_g^w\{f\} \tag{2.22}$$

where $f, \mathfrak{D}_{w,g}\,f \in L^1_{w,g}$.

Proof Using integration by parts, we acquire

$$\begin{aligned}\mathcal{F}_g^w\{\mathfrak{D}_{w,g}\,f\} &= \int_{-\infty}^{\infty}\left[\frac{1}{w(x)}\frac{1}{g'(x)}D_x\,(w(x)f(x))\right]e^{-ikg(x)}\,w(x)g'(x)dx\\ &= w(x)f(x)\,e^{-ikg(x)}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} w(x)f(x)\,(-ik)e^{-ikg(x)}\,g'(x)\,dx\\ &= ik\,\mathcal{F}_g^w\{f\}\end{aligned}$$

thanks to $\lim_{x\to\pm\infty} w(x)f(x) = 0$. □

To avoid notational confusion when taking the weighted transform of the n-th order w-weighted derivative of a function $f \in L^1_{w,g}$ with respect to a function g, we define the following function space

$$U^n_{w,g} := \left\{ f \in L^1_{w,g} \,\middle|\, \mathfrak{D}^j_{w,g}\,f \in L^1_{w,g},\ \lim_{|x|\to\infty} f^*_{j-1}(x) = 0,\ j = 1, 2, \ldots, n \right\} \tag{2.23}$$

where $n \in \mathbb{N}$ and

$$f^*_{j-1}(x) = \left(\frac{1}{g'(x)}D_x\right)^{j-1}(w(x)f(x)), \quad j \in \mathbb{N}. \tag{2.24}$$

Theorem 2.15 *If* $f \in U^n_{w,g}$, *then*

$$\mathcal{F}_g^w\{\mathfrak{D}^n_{w,g}\,f\} = (ik)^n\,\mathcal{F}_g^w\{f\}, \quad n \in \mathbb{N}. \tag{2.25}$$

Proof This result follows immediately from Theorem 2.14 by induction on n. □

Lemma 2.16 *We have*

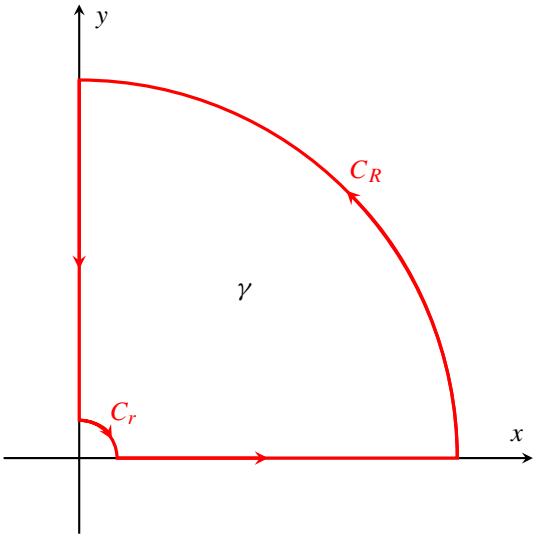
$$\int_0^{\infty} \frac{e^{-iay}dy}{y^{\beta}} = (ia)^{\beta-1}\,\Gamma(1-\beta) \tag{2.26}$$

where $0 < Re(\beta) < 1$ *and* $a \in \mathbb{R} \setminus \{0\}$.

Proof From the definition of the Gamma function, we get

$$\Gamma(1-\beta)=\int_0^{\infty} e^{-t}\, t^{-\beta}\, dt, \quad \mathrm{Re}(\beta)<1. \tag{2.27}$$

Let the contour γ be as follow:



So, we have

$$\int_{\gamma} \frac{e^{-z}dz}{z^{\beta}}=0 \tag{2.28}$$

from Cauchy–Goursat theorem in [17] because there are no singularities inside the contour γ. We clearly acquire the following equality

$$\int_{\gamma} \frac{e^{-z}dz}{z^{\beta}}=\int_r^R \frac{e^{-x}dx}{x^{\beta}}+\int_{C_r} \frac{e^{-z}dz}{z^{\beta}}+\int_R^r \frac{e^{-iy}d(iy)}{(iy)^{\beta}}+\int_{C_R} \frac{e^{-z}dz}{z^{\beta}}. \tag{2.29}$$

If we take the limit for $r\to 0$ and $R\to\infty$ in the equation (2.29), we get

$$0=\int_0^{\infty} \frac{e^{-x}dx}{x^{\beta}}+\int_{\infty}^0 \frac{e^{-iy}d(iy)}{(iy)^{\beta}} \tag{2.30}$$

from the equality in (2.28). Thus it is that

$$\int_0^{\infty} \frac{e^{-iy}dy}{y^{\beta}}=i^{\beta-1}\int_0^{\infty} \frac{e^{-x}dx}{x^{\beta}} \quad \text{and} \quad \int_0^{\infty} \frac{e^{-iay}dy}{y^{\beta}}=(ia)^{\beta-1}\,\Gamma(1-\beta) \tag{2.31}$$

where $0<\mathrm{Re}(\beta)<1$ thanks to the equation in (2.27). □

Theorem 2.17 *Let $\alpha \in \mathbb{R}$, $n-1 < \alpha \leq n$ and $f \in U^n_{w,g}$. Thus, the following equality holds*

$$\mathcal{F}^w_g\{\mathfrak{C}^\alpha_{w,g} f\} = \big(ik\big)^\alpha \mathcal{F}^w_g\{f_+\}, \quad k \neq 0, \tag{2.32}$$

where $f_+(t) = u(t) f(t)$.

Proof If we choose $h(t)$ as

$$h(t) = \frac{u\Big(x - g^{-1}\big[g(x) - g(t)\big]\Big)\big[g(t)\big]^{n-\alpha-1}}{\Gamma(n-\alpha)\, w(t)} \tag{2.33}$$

in the definition of the weighted convolution in the equality (2.18), we acquire

$$\mathfrak{C}^\alpha_{w,g}\, f = h \ *^w_g \ \mathfrak{D}^n_{w,g}\, f_+ \tag{2.34}$$

due to the equality in (1.9) where $f_+(t) = u(t) f(t)$. Now, let's calculate the weighted Fourier transform of the function h. It follows that

$$\begin{aligned}
\mathcal{F}^w_g\{h\} &= \int_{-\infty}^{\infty} \frac{u\Big(x - g^{-1}\big[g(x) - g(t)\big]\Big)\big[g(t)\big]^{n-\alpha-1}}{\Gamma(n-\alpha)\, w(t)}\, e^{-ikg(t)}\, w(t) g'(t)\, dt \\
&= \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^{\infty} u\Big(x - g^{-1}\big[g(x) - g(t)\big]\Big)\big[g(t)\big]^{n-\alpha-1}\, e^{-ikg(t)}\, g'(t)\, dt \\
&= \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^{\infty} u\Big(x - g^{-1}\big[g(x) - z\big]\Big) z^{n-\alpha-1}\, e^{-ikz}\, dz
\end{aligned} \tag{2.35}$$

by substituting $z = g(t)$. Note that $g' > 0$ and $\lim_{t\to\pm\infty} g(t) = \pm\infty$.

- If $z = 0$ then $x - g^{-1}\big[g(x) - z\big] = 0$,
- If $z > 0$ then $x - g^{-1}\big[g(x) - z\big] > 0$,
- If $z < 0$ then $x - g^{-1}\big[g(x) - z\big] < 0$.

Therefore, it holds that

$$u(z) = u\Big(x - g^{-1}\big[g(x) - z\big]\Big). \tag{2.36}$$

We derive

$$\begin{aligned}
\mathcal{F}^w_g\{h\} &= \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^{\infty} u(z) z^{n-\alpha-1}\, e^{-ikz}\, dz \\
&= \frac{1}{\Gamma(n-\alpha)} \int_{0}^{\infty} z^{n-\alpha-1}\, e^{-ikz}\, dz \\
&= \big(ik\big)^{\alpha-n}, \quad k \neq 0
\end{aligned} \tag{2.37}$$

using the equation in (2.36) and Lemma 2.16. Eventually, we obtain

$$\begin{aligned}\mathcal{F}_g^w\{\mathfrak{C}_{w,g}^{\alpha} f\} &= \mathcal{F}_g^w\{h \ *_g^w \ \mathfrak{D}_{w,g}^n f_+\} \\ &= \mathcal{F}_g^w\{h\}\, \mathcal{F}_g^w\{\mathfrak{D}_{w,g}^n f_+\} \\ &= (ik)^{\alpha-n}(ik)^n \mathcal{F}_g^w\{f_+\} \\ &= (ik)^{\alpha} \mathcal{F}_g^w\{f_+\} \end{aligned} \tag{2.38}$$

from the equality in (2.34) and, Theorems 2.13 and 2.15. □

Theorem 2.18 *Let $f, \mathfrak{J}_{w,g}^{\alpha} f \in L_{w,g}^1$, then the following holds.*

$$\mathcal{F}_g^w\{\mathfrak{J}_{w,g}^{\alpha} f\} = (ik)^{-\alpha} \mathcal{F}_g^w\{f_+\}, \quad k \neq 0, \tag{2.39}$$

where $\alpha \in \mathbb{R}$ and $f_+(t) = u(t) f(t)$.

Proof If we set

$$h(t) = \frac{u\Big(x - g^{-1}\big[g(x) - g(t)\big]\Big)\big[g(t)\big]^{\alpha-1}}{\Gamma(\alpha)\, w(t)} \tag{2.40}$$

in the equality (2.18), we have

$$\mathfrak{J}_{w,g}^n f = h \ *_g^w \ f_+ \tag{2.41}$$

thanks to the equality in (1.10) where $f_+(t) = u(t) f(t)$. By performing similar operations in the proof of Theorem 2.17, we have

$$\mathcal{F}_g^w\{h\} = (ik)^{-\alpha}, \quad k \neq 0. \tag{2.42}$$

There exists

$$\begin{aligned}\mathcal{F}_g^w\{\mathfrak{J}_{w,g}^{\alpha} f\} &= \mathcal{F}_g^w\{h \ *_g^w \ f_+\} \\ &= \mathcal{F}_g^w\{h\}\, \mathcal{F}_g^w\{f_+\} \\ &= (ik)^{-\alpha} F_g^w\{f_+\} \end{aligned} \tag{2.43}$$

when the equalities in (2.41) and (2.42) and Theorem 2.13 are considered together. □

Theorem 2.19 *Let $\alpha \in \mathbb{R}$ and $n - 1 < \alpha \leq n$. So, for $\mathfrak{J}_{w,g}^{\alpha} f \in U_{w,g}^n$, we have*

$$\mathcal{F}_g^w\{\mathfrak{R}_{w,g}^{\alpha} f\} = (ik)^{\alpha} \mathcal{F}_g^w\{f_+\}, \quad k \neq 0, \tag{2.44}$$

where $f_+(t) = u(t) f(t)$.

Proof Considering Theorems (2.15) and (2.18) together, we obtain

$$\mathcal{F}_g^w\{\mathfrak{R}_{w,g}^{\alpha}\, f\} = \mathcal{F}_g^w\{\mathcal{D}_{w,g}^n\, \left(\mathfrak{I}_{w,g}^{n-\alpha}\, f\right)\} = \big(ik\big)^n\, \mathcal{F}_g^w\{\mathfrak{I}_{w,g}^{n-\alpha}\} = \big(ik\big)^{\alpha}\, \mathcal{F}_g^w\{f_+\} \quad (2.45)$$

due to $\mathcal{F}_g^w\{\mathfrak{I}_{w,g}^{n-\alpha}\} = \big(ik\big)^{\alpha-n}\, \mathcal{F}_g^w\{f_+\}$. □

3 Some Examples

Example 3.1 Let us assume $0 < \alpha < 1$ and

$$f(x) = \begin{cases} 0, & x < 0 \\ x^2 e^{-x^4/2}, & x \geqslant 0 \end{cases} \tag{3.1}$$

and let us take

$$w(x) = 1 \quad \text{and} \quad g(x) = x^2 \text{ for } x \geqslant 0. \tag{3.2}$$

So, we obtain

$$\mathfrak{D}_{w,g}^1\, f(x) = \begin{cases} 0, & x < 0 \\ \big(1 - x^4\big)e^{-x^4/2}, & x \geqslant 0 \end{cases}, \tag{3.3}$$

$$\mathfrak{C}_{w,g}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{\big(1 - t^4\big)e^{-t^4/2}}{\big(x^2 - t^2\big)^{\alpha}}\, 2t\, dt, \tag{3.4}$$

$$\mathfrak{I}_{w,g}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{t^2 e^{-t^4/2}}{\big(x^2 - t^2\big)^{1-\alpha}}\, 2t\, dt, \quad \mathfrak{I}_{w,g}^{1} f(x) = 1 - e^{-x^4/2}. \tag{3.5}$$

Weighted Fourier transforms of $f(x)$, $\mathfrak{C}_{w,g}^{\alpha} f(x)$ and $\mathfrak{I}_{w,g}^{\alpha} f(x)$ are

$$F(k) = \mathcal{F}_g^w\{f\} = \int_0^{\infty} x^2 e^{-x^4/2}\, e^{-ikx^2}\, 2x\, dx, \tag{3.6}$$

$$F_{\alpha}(k) = \mathcal{F}_g^w\{\mathfrak{C}_{w,g}^{\alpha} f(x)\} = \frac{1}{\Gamma(1-\alpha)} \int_0^{\infty} \int_0^x \frac{\big(1 - t^4\big)e^{-t^4/2}}{(x^2 - t^2)^{\alpha}}\, e^{-ikx^2} 2t\, 2x\, dt\, dx, \tag{3.7}$$

$$G_{\alpha}(k) = \mathcal{F}_g^w\{\mathfrak{I}_{w,g}^{\alpha} f(x)\} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \int_0^x \frac{t^2 e^{-t^4/2}}{\big(x^2 - t^2\big)^{1-\alpha}}\, e^{-ikx^2} 2t\, 2x\, dt\, dx, \tag{3.8}$$

respectively. Let's proceed with verifying Theorem 2.17 for this example:

If change order of integration in (3.7), we have

$$F_\alpha(k) = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \int_t^\infty \frac{\left(1-t^4\right)e^{-t^4/2}}{(x^2-t^2)^\alpha} e^{-ikx^2}\, 2t\, 2x\, dx\, dt. \tag{3.9}$$

By using substitution $u^2 = x^2 - t^2$ in (3.9), we obtain

$$F_\alpha(k) = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \int_0^\infty \frac{\left(1-t^4\right)e^{-t^4/2}}{u^{2\alpha}} e^{-ik(u^2+t^2)}\, 2t\, 2u\, du\, dt. \tag{3.10}$$

The integral in (3.10) can be written as two separate integrals the following that

$$F_\alpha(k) = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \frac{e^{-iku^2}}{u^{2\alpha}} 2u\, du \int_0^\infty \left(1-t^4\right)e^{-t^4/2}\, e^{-ikt^2}\, 2t\, dt. \tag{3.11}$$

Using

$$\int_0^\infty \frac{e^{-iku^2}}{u^{2\alpha}} 2u\, du = \int_0^\infty \frac{e^{-ikz}}{z^\alpha}\, dz = \frac{\Gamma(1-\alpha)}{(ik)^{1-\alpha}}, \quad 0 < \alpha < 1, \quad k \neq 0, \tag{3.12}$$

we acquire

$$F_\alpha(k) = \frac{1}{(ik)^{1-\alpha}} \int_0^\infty \left(1-t^4\right)e^{-t^4/2}\, e^{-ikt^2}\, 2t\, dt. \tag{3.13}$$

Using integration by parts where $u = e^{-ikt^2}$ and $dv = \left(1-t^4\right)e^{-t^4/2}\, 2t\, dt$, we obtain

$$\begin{aligned}\int_0^\infty \left(1-t^4\right)e^{-t^4/2}\, e^{-ikt^2}\, 2t\, dt &= e^{-ikt^2} t^2 e^{-t^4/2}\Big|_0^\infty - \int_0^\infty t^2 e^{-t^4/2}\, (-2ikt)\, e^{-ikt^2}\, dt \\ &= (ik) \int_0^\infty t^2 e^{-t^4/2}\, 2t\, e^{-ikt^2}\, dt. \end{aligned} \tag{3.14}$$

Finally, by writing (3.14) in (3.13), we get the following equality

$$F_\alpha(k) = (ik)^\alpha \int_0^\infty t^2 e^{-t^4/2}\, e^{-ikt^2}\, 2t\, dt = (ik)^\alpha F(k). \tag{3.15}$$

Let's proceed with verifying Theorem 2.18 for this example:
When we change the order of integration in (3.8), we get

$$G_\alpha(k) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_t^\infty \frac{t^2 e^{-t^4/2}}{(x^2-t^2)^{1-\alpha}} e^{-ikx^2}\, 2t\, 2x\, dx\, dt. \tag{3.16}$$

Using substitution $u^2 = x^2 - t^2$ in (3.16), we have

$$G_\alpha(k) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_0^\infty \frac{t^2 e^{-t^4/2}}{u^{2-2\alpha}} e^{-ik(u^2+t^2)}\, 2t\, 2u\, du\, dt. \tag{3.17}$$

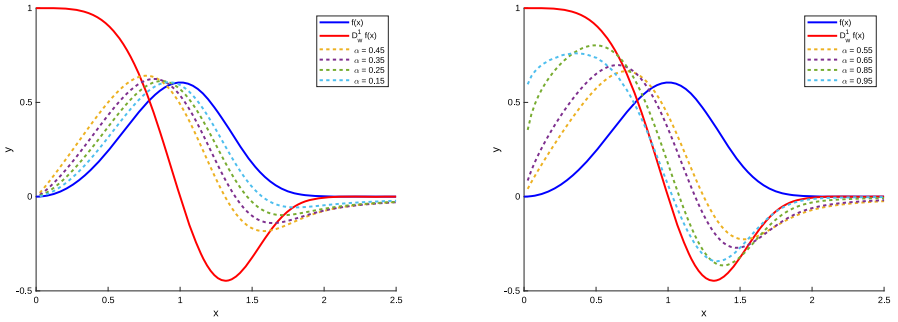


Fig. 1 Graphs of weighted fractional and integer order derivatives, $w(x) = 1$ and $g(x) = x^2$, for the function given in (3.1)

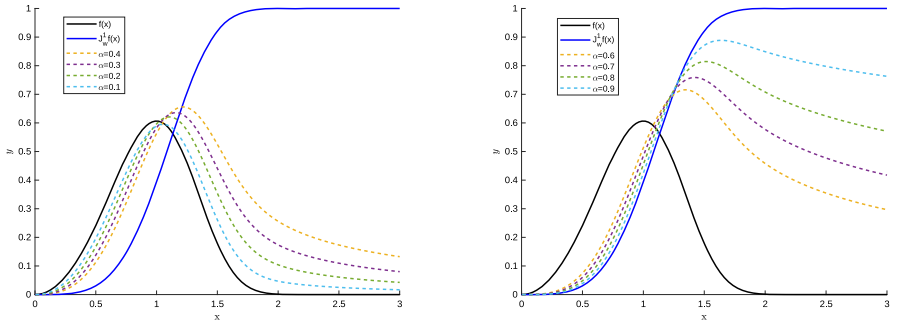


Fig. 2 Graphs of weighted fractional and integer order integrals, with $w(x) = 1$ and $g(x) = x^2$, for the function given in (3.1)

By rewriting the integral in (3.17) as two separate integrals, and performing the necessary steps, we have

$$\begin{aligned} G_\alpha(k) &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{e^{-iku^2}}{u^{2-2\alpha}}\, 2u\, du \int_0^\infty t^2 e^{-t^4/2}\, e^{-ikt^2}\, 2t\, dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{e^{-ikz}}{z^{1-\alpha}}\, dz\, F(k) \\ &= \frac{1}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(ik)^\alpha}\, F(k) \\ &= (ik)^{-\alpha} F(k). \end{aligned} \tag{3.18}$$

For function f given in (3.1), in Fig. 1, the graphs of weighted integer order derivative $\mathfrak{D}^1_{w,g} f(x)$ and the fractional derivative $\mathfrak{C}^\alpha_{w,g} f(x)$, while in Fig. 2, the weighted integer order integral $\mathfrak{J}^1_{w,g} f(x)$ and the fractional integer $\mathfrak{J}^\alpha_{w,g} f(x)$ are plotted using numerical integration for some α values, where $w(x) = 1$, $g(x) = x^2$ and $x \geq 0$.

Example 3.2 Let us assume $0 < \alpha < 1$ and

$$h(x) = \begin{cases} 0, & x < 0 \\ 12x^5 - 45x^4 + 40x^3, & x \geqslant 0 \end{cases} \tag{3.19}$$

and let us take

$$w(x) = e^{-|x|} \quad \text{and} \quad g(x) = x^3. \tag{3.20}$$

Thus, we achieve

$$\mathfrak{D}^1_{w,g}\, h(x) = \begin{cases} 0, & x < 0 \\ -4x^3 + 35x^2 - \frac{220}{3}x + 40, & x \geqslant 0 \end{cases}, \tag{3.21}$$

$$\mathfrak{C}^{\alpha}_{w,g} h(x) = \frac{e^x}{\Gamma(1-\alpha)} \int_0^x \left(-4t^3 + 35t^2 - \frac{220}{3}t + 40\right) \frac{e^{-t}\, 3t^2\, dt}{\left(x^3 - t^3\right)^{\alpha}}. \tag{3.22}$$

Weighted Fourier transforms of $h(x)$ and $\mathfrak{C}^{\alpha}_{w,g} h(x)$ are

$$H(k) = \mathcal{F}^w_g\{h\} = \int_0^{\infty} e^{-x}\left(12x^5 - 45x^4 + 40x^3\right) e^{-ikx^3}\, 3x^2\, dx, \tag{3.23}$$

$$H_\alpha(k) = \frac{1}{\Gamma(1-\alpha)} \int_0^{\infty} \left(\int_0^x \left(-4t^3 + 35t^2 - \frac{220}{3}t + 40\right) \frac{e^{-t}\, e^{-ikx^3}}{\left(x^3 - t^3\right)^{\alpha}}\, 3t^2\, dt \right) 3x^2\, dx \tag{3.24}$$

where $H_\alpha(k) = \mathcal{F}^w_g\{\mathfrak{C}^{\alpha}_{w,g} h(x)\}$, respectively. Let's proceed with verifying Theorem 2.17 for this example:

If change order of integration in (3.24), we have

$$H_\alpha(k) = \frac{1}{\Gamma(1-\alpha)} \int_0^{\infty} \left(\int_t^{\infty} \left(-4t^3 + 35t^2 - \frac{220}{3}t + 40\right) \frac{e^{-t}\, e^{-ikx^3}}{\left(x^3 - t^3\right)^{\alpha}}\, 3x^2\, dx \right) 3t^2\, dt. \tag{3.25}$$

By using substitution $u^3 = x^3 - t^3$ in (3.25), we have

$$H_\alpha(k) = \frac{1}{\Gamma(1-\alpha)} \int_0^{\infty} \left(\int_0^{\infty} \left(-4t^3 + 35t^2 - \frac{220}{3}t + 40\right) \frac{e^{-t}\, e^{-ik(u^3+t^3)}}{u^{3\alpha}}\, 3u^2\, du \right) 3t^2\, dt. \tag{3.26}$$

This integral in (3.26) can be written as two separate integrals as follows

$$H_\alpha(k) = \frac{1}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{e^{-iku^3}}{u^{3\alpha}} 3u^2\, du \int_0^{\infty} \left(-4t^3 + 35t^2 - \frac{220}{3}t + 40\right) e^{-t}\, e^{-ikt^3}\, 3t^2\, dt. \tag{3.27}$$

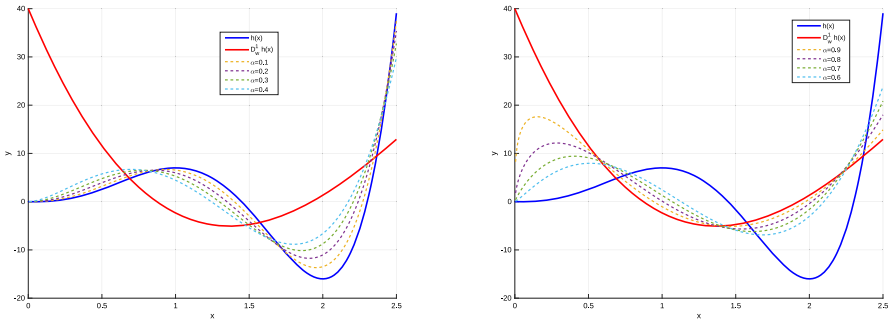


Fig. 3 Graphs of weighted fractional and integer order derivatives of the function in (3.19), with $w(x) = e^{-|x|}$ and $g(x) = x^3$

By using

$$\int_0^\infty \frac{e^{-iku^3}}{u^{3\alpha}} 3u^2\, du = \int_0^\infty \frac{e^{-ikz}}{z^\alpha}\, dz = \frac{\Gamma(1-\alpha)}{(ik)^{1-\alpha}}, \quad 0 < \alpha < 1, \quad k \neq 0, \tag{3.28}$$

we have

$$H_\alpha(k) = \frac{1}{(ik)^{1-\alpha}} \int_0^\infty \left(-4t^3 + 35t^2 - \frac{220}{3}t + 40\right) e^{-t}\, e^{-ikt^3}\, 3t^2\, dt. \tag{3.29}$$

Using integration by parts where

$$u = e^{-ikt^3} \quad \text{and} \quad dv = \left(-4t^3 + 35t^2 - \frac{220}{3}t + 40\right) e^{-t}\, 3t^2\, dt, \tag{3.30}$$

we arrive at

$$\begin{aligned} H_\alpha(k) &= \frac{1}{(ik)^{1-\alpha}} \left[e^{-ikt^3}\big(12t^2 - 45t + 40\big)t^3 e^{-t}\Big|_0^\infty \right. \\ &\quad \left. + ik \int_0^\infty \big(12t^2 - 45t + 40\big)t^3 e^{-t} e^{-ikt^3} 3t^2\, dt \right] \\ &= \frac{1}{(ik)^{1-\alpha}} \left[ik \int_0^\infty \big(12t^2 - 45t + 40\big)t^3 e^{-t} e^{-ikt^3} 3t^2\, dt \right] \\ &= (ik)^\alpha \int_0^\infty e^{-t}\big(12t^5 - 45t^4 + 40t^3\big) e^{-ikt^3} 3t^2\, dt \\ &= (ik)^\alpha H(k). \end{aligned} \tag{3.31}$$

For function h given in (3.19), in Fig. 3, the graphs of weighted integer order derivative $\mathfrak{D}^1_{w,g} h$ and the fractional derivative $\mathfrak{C}^\alpha_{w,g} h$ with $w(x) = e^{-|x|}$ and $g(x) = x^3$ are displayed.

4 Conclusion

In this research, a weighted generalization of the Fourier transform is defined and developed. Firstly, suitable normed spaces are introduced, and the existence of the weighted Fourier transform in these normed spaces is proven. Then, the inverse of the weighted Fourier transform of a function is formulated. Since the weighted Fourier transform of the product of two functions is not the same as the product of the individual weighted Fourier transforms of these functions, a convolution for weighted Fourier transform is defined, and some properties are obtained. Among the most important results of this study are the properties provided by the weighted Fourier transforms of weighted derivative and weighted fractional derivative of a function f with a weighted w respect to another function g. These results can be used as a tool in solving some weighted differential (or weighted fractional differential) ordinary or partial differential equations.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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