Cyclic (α,β)-Admissible Mappings in Modular Spaces and Applications to Integral Equations

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Abstract

The main concern of this study is to present a generalization of Banach's fixed point theorem in some classes of modular spaces, where the modular is convex and satisfying the Δ_2-condition. In this work, the existence and uniqueness of fixed point for $(\alpha,\beta)-(\psi,\varphi)$-contractive mapping and weak $\alpha-\beta-\psi$- rational contraction in modular spaces are proved. Some examples are supplied to support the usability of our results. As an application, the existence of a solution for an integral equation of Lipschitz type in a Musielak-Orlicz space are presented.

1. Introduction and Preliminaries

It is well known fixed point theorems have important roles and applications in mathematics analysis, particularly in differential and integral equations. One most popular fixed point theorem is Banach Fixed Point Theorem [6]. By using this theorem, most authors have proved several fixed point theorems for various mappings. Such as, Dutta and Choudhury proved (ψ,ϕ)- contractive mappings in complete metric space [11]. Samet et al. introduced the concept of $\alpha-\varphi$- contractive type mappings and established various fixed point theorems [31]. Recently, Salimi et al. modified the concept of $\alpha-\varphi$- contractive type mappings [30]. Later, Alizadeh et al. [4] developed a new fixed point theorem in complete metric spaces. They introduced the concept of cyclic $(\alpha,\beta)-$admissible mappings and $(\alpha,\beta)-(\psi,\phi)$-contractive mapping and established some fixed point results in metric spaces.
On the other hand, some authors introduced a new concept of modular vector spaces which are natural generalizations of many classical function spaces. Firstly, Nakano initiated the concept of modulared spaces [25]. Later, some authors proved new fixed point theorems of Banach type in modular spaces [1, 13, 16, 20, 21, 32, 2, 3, 5, 8, 9, 10, 12, 14, 19, 23, 28, 29, 33].
In this work, we present some fixed point results as a generalization of Banach's fixed point theorem using some convenient constants in the contraction assumption in modular spaces. Motivated by [4] and [24], we prove some fixed point results for $(\alpha,\beta)-(\psi,\phi)$-contractive mappings in modular spaces. Some examples are supplied in order to support the useability of our results. As an application we show that the existence and uniqueness of solutions for an integral equation of Lipschitz type in a Musielak-Orlicz space.

Definition 1.1. *[24, 26] Let X be an arbitrary vector space. A functional $\rho: X\to[0,\infty)$ is called a modular if, for any x,y in X, the following conditions hold:*

(a) $\rho(x)=0$ *if and only if* $x=0$,
(b) $\rho(ax)=\rho(x)$ *for every scalar* α *with* $|\alpha|=1$,
(c) $\rho(\alpha x+\beta y)\le\rho(x)+\rho(y)$, *whenever* $\alpha+\beta=1$, *and* $\alpha,\beta\ge 0$.

If (c) is replaced with $\rho(\alpha x+\beta y)\le\alpha^s\rho(x)+\beta^s\rho(y)$ *where* $\alpha^s+\beta^s=1$, $\alpha,\beta\ge 0$, *and* $s\in(0,1]$, *then* ρ *is called s-convex modular. If* $s=1$, *then we say that* ρ *is convex modular. The following are some consequences of condition (c).*

Remark 1.2. *[7]*

(a) For $a,b\in R$ *with* $|a|<|b|$ *we have* $\rho(ax)<\rho(bx)$ *for all* $x\in X$.

(b) For $a_1,\dots,a_n\in R+$ *with* $\sum_{i=1}^n a_i=1$, *we have*

$$\rho\Big(\sum_{i=1}^n a_ix_i\Big)=\rho\Big(\sum_{i=1}^n x_i\Big)\ \text{for any } x_1,\dots,x_n\in X.$$

Remark 1.3. *[25] A modular ρ defines a corresponding modular space, i.e. the space X given by*

$$X_\rho=\{x\in X;\rho(\lambda x)\to 0\ as\ \lambda\to 0\}.$$

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Definition 1.4. *[17] A sequence $\{x_n\}$ in modular space X_ρ is said to be:*

(a) ρ-convergent to $x \in X_\rho$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.

(b) ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.

(c) X_ρ is called ρ-complete if any ρ-Cauchy sequence is ρ-convergent.

(d) ρ is said to be Δ_2-condition if $\rho(2x_n) \to 0$ as $n \to \infty$, whenever $\rho(x_n) \to 0$ as $n \to \infty$.

Definition 1.5. *[4] Let $T : X \to X$ be a mapping and $\alpha, \beta : X \to R_+$ be two functions. We say that T is a cyclic (α, β)-admissible mapping if*

(i) $\alpha(x) \geq 1$ for some $x \in X$ implies $\beta(Tx) \geq 1$,
(ii) $\beta(x) \geq 1$ for some $x \in X$ implies $\alpha(Tx) \geq 1$.

Definition 1.6. *[4] Let be Ψ the set of continuous and increasing functions $\psi : [0,\infty) \to [0,\infty)$ and Φ be the set of lower semicontinuous functions $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(t) = 0$ iff $t = 0$. Let X be a metric space and $T : X \to X$ be a cyclic (α, β)-admissible mapping. We say that T is a $(\alpha, \beta) - (\psi, \phi)-$contractive mapping if*

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)) \tag{1.1}$$

for $x, y \in X$, where $\psi \in \Psi$ and $\phi \in \Phi$.

Theorem 1.7. *[17] Let C be a ρ-complete, ρ-bounded subset of L_ρ and let $T : C \to C$ be a ρ-strict contraction. Then T has a unique fixed point $z \in C$. Moreover, z is the ρ-limit of the iterate of any point in C under the action of T.*

2. Main Results

We denote by Ψ the set of continuous and increasing functions $\psi : [0,\infty) \to [0,\infty)$ and denote by Φ the set of lower semicontinuous functions $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(t) = 0$ iff $t = 0$. Let X_ρ be a nonempty set and let $T : X_\rho \to X_\rho$ be an arbitrary mapping. We say that $x \in X_\rho$ is a fixed point for T, if $x = Tx$. We denote by $Fix(T)$ the set of all fixed points of T. In the sequel, suppose the modular ρ is convex and satisfies the Δ_2-condition.

Definition 2.1. *Let be Ψ the set of continuous and increasing functions $\psi : [0,\infty) \to [0,\infty)$ and Φ be the set of lower semicontinuous functions $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(t) = 0$ iff $t = 0$. Let X_ρ be a ρ-complete modular space and $T : X \to X$ be a cyclic (α, β)-admissible mapping. We say that T is a $(\alpha, \beta) - (\psi, \phi)-$contractive mapping if*

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \psi(\rho(Tx, Ty)) \leq \psi(\rho(x, y)) - \phi(\rho(x, y)) \tag{2.1}$$

for $x, y \in X$, where $\psi \in \Psi$ and $\phi \in \Phi$.

Now we are ready to prove our new first result.

Theorem 2.2. *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be a $(\alpha, \beta) - (\psi, \phi)$-contravtive mapping. Suppose that the following conditions hold:*

(a) there exists $x_0 \in X_\rho$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,

(b) T is continuous, or

(c) if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$,

then T has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Proof. Define a sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in N$. Since T is a cyclic (α, β)-admissible mapping and $\alpha(x_0) \geq 1$ then $\beta(x_1) = \beta(Tx_0) \geq 1$ which implies $\alpha(x_2) = \alpha(Tx_1) \geq 1$. By continuing this process, we get $\alpha(x_{2n}) \geq 1$ and $\beta(x_{2n-1}) \geq 1$ for all $n \in N$. Again, since T is a cyclic (α, β)-admissible mapping and $\beta(x_0) \geq 1$, by the similar method, we have $\beta(x_{2n}) \geq 1$ and $\alpha(x_{2n-1}) \geq 1$ for all $n \in N$. That is, $\alpha(x_n) \geq 1$ and $\beta(x_n) \geq 1$ for all $n \in N \cup \{0\}$. Equivalently, $\alpha(x_{n-1})\beta(x_n) \geq 1$ for all $n \in N$. Therefore by (2.1), we have

$$\begin{aligned} \psi(\rho(x_n - x_{n+1})) &\leq \psi(\rho(x_{n-1} - x_n)) - \phi(\rho(x_{n-1} - x_n)) \\ &\leq \psi(\rho(x_{n-1} - x_n)) \end{aligned} \tag{2.2}$$

and since ψ is increasing, we get

$$\rho(x_n - x_{n+1}) \leq \rho(x_{n-1} - x_n)$$

for all $n \in N$. So, $\{\rho_n := \rho(x_n - x_{n+1})\}$ is a non-increasing sequence of positive real numbers. Then there exist $r \geq 0$ such that $\lim_{n\to\infty} \rho_n = r$. We shall show that $r = 0$. By taking the limsup on both sides of (2.2), we have

$$\lim_{n\to\infty} \rho(x_n - x_{n+1}) = 0. \tag{2.3}$$

Now, we want to show that $\{x_{2n}\}$ is a Cauchy sequence. Suppose to the contrary, that $\{x_{2n}\}$ is not a Cauchy sequence. Then there are $\varepsilon > 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, $n(k) > m(k) > k$,

$$\rho(x_{2n(k)} - x_{2m(k)}) \geq \varepsilon \text{ and } \rho(2(x_{2n(k)-1} - x_{2m(k)})) < \varepsilon. \tag{2.4}$$

Now for all $k \in N$, we have

$$\begin{aligned} \varepsilon &\leq \rho(x_{2n(k)} - x_{2m(k)}) \\ &\leq \rho(2(x_{2n(k)} - x_{2n(k)-1})) + \rho(2(x_{2n(k)-1} - x_{2m(k)})) \\ &< \rho(2(x_{2n(k)} - x_{2n(k)-1})) + \varepsilon. \end{aligned}$$

Taking the limit as $k \to +\infty$ in the above inequality and using (2.3), we get

$$\lim_{k\to\infty} \rho(x_{2n(k)} - x_{2m(k)}) = \varepsilon. \tag{2.5}$$

Since

$$\begin{aligned} \rho(x_{2n(k)+1} - x_{2m(k)+1}) &= \rho(x_{2n(k)+1} - x_{2n(k)} + x_{2n(k)} - x_{2m(k)+1}) \\ &\leq \rho(2(x_{2n(k)+1} - x_{2n(k)})) + \rho(2(x_{2n(k)} - x_{2m(k)+1})) \end{aligned}$$

and

$$\rho(2(x_{2n(k)} - x_{2m(k)+1})) = \rho(2(x_{2n(k)} - x_{2m(k)} + x_{2m(k)} - x_{2m(k)+1})) \leq \rho(4(x_{2n(k)} - x_{2m(k)}) + \rho(4(x_{2m(k)} - x_{2m(k)+1}))$$

then by taking the limit as $k \to +\infty$ in above inequality and using (2.3) and (2.5), we deduce that

$$\lim_{k\to\infty} \rho(x_{2n(k)+1} - x_{2m(k)+1}) = \varepsilon. \tag{2.6}$$

Now, by (2.1), we get

$$\psi(\rho(x_{2n(k)+1} - x_{2m(k)+1})) \leq \psi(\rho(x_{2n(k)} - x_{2m(k)})) - \phi(\rho(x_{2n(k)} - x_{2m(k)})). \tag{2.7}$$

Since, $\alpha(x_{2n(k)})\beta(x_{2m(k)}) \geq 1$ for all $k \in N$. By taking the limsup on both sides of (2.7) and applying (2.4) and (2.6), we obtain

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon)$$

That is, $\varepsilon = 0$ which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since X_ρ is a complete modular space, then there is $z \in X$ such that $x_n \to z$ as $n \to \infty$. First, we assume that T is continuous. Hence, we deduce

$$Tz = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z.$$

So z is a fixed point of T . Now, assume that (c) is held, That is, $\alpha(x_n)\beta(z) \geq 1$. From (2.1) we have

$$\psi(\rho(x_{n+1} - Tz)) \leq \psi(\rho(x_n - z)) - \phi(\rho(x_n - z)). \tag{2.8}$$

By taking the limsup on both sides of (2.8), we get $\psi(\rho(z - Tz)) = 0$. Then $\rho(z - Tz) = 0$. i.e., $z = Tz$. To prove the uniqueness of fixed point, suppose that z and z^* are two fixed points of T . Since $\alpha(z)\beta(z^*) \geq 1$, it follows from (1) that

$$\psi(\rho(z - Tz^*)) \leq \psi(\rho(z - z^*)) - \phi(\rho(z - z^*)).$$

So $\phi(\rho(z - z^*)) = 0$ and hence $\rho(z - z^*) = 0$ i.e., $z = z^*$. □

Corollary 2.3. *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be a cyclic (α, β)-admissible mapping such that*

$$\alpha(x)\beta(y)\psi(\rho(Tx - Ty)) \leq \psi(\rho(x - y)) - \phi(\rho(x - y))$$

for all $x, y \in X_\rho$ where $\psi \in \Psi$ and $\phi \in \Phi$. Suppose that the following assertions hold:

(a) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,

(b) T is continuous, or

(c) if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$,

then T has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Proof. Let $\alpha(x)\beta(y) \geq 1$ for $x, y \in X$. Then by (2.8), we have

$$\psi(\rho(Tx - Ty)) \leq \psi(\rho(x - y)) - \phi(\rho(x - y)).$$

This implies that the inequality (2.1) holds. Therefore, the proof follows from Theorem 2.2. □

Remark 2.4. *Corallary 2.3 implies Theorem 1.7. Indeed, if for each $f, g, t \in X_\rho$, $\alpha(f)\beta(g) = 1$, $\psi(t) = t$, $\phi(t) = kt$ for $k \in (0, 1)$ in Theorem 2.2, we have the following ρ-strict contraction*

$$\rho(Tf - Tg)) \leq k\rho(f - g).$$

Corollary 2.5. *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be a cyclic (α,β)-admissible mapping such that*

$$(\alpha(x)\beta(y)+1)^{\psi(\rho(fx-fy))} \leq 2^{\psi(\rho(x-y))-\phi(\rho(x-y))}$$

for all $x,y \in X_\rho$ where $\psi \in \Psi$ and $\phi \in \Phi$. Suppose that the following assertions hold:

(a) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,

(b) T is continuous, or

(c) if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$,

then T has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x,y \in Fix(T)$, then T has a unique fixed point.

Example 2.6. *Let $X_\rho = R$, $\rho(x) = |x|$ for all $x \in X_\rho$, and $T : X_\rho \to X_\rho$ by*

$$Tx = \begin{cases} \frac{x^2}{3} & , \quad x \in [-2,2] \\ \sqrt{x} & , \quad otherwise. \end{cases}$$

Define $\psi,\phi : [0,+\infty) \to [0,+\infty)$ by $\psi(t) = 3t, \phi(t) = t$ and $\alpha,\beta : X_\rho \to [0,+\infty)$ by

$$\alpha(x) = \begin{cases} 1 & , \quad x \in [-2,\frac{4}{3}] \\ 0 & , \quad otherwise. \end{cases}$$

and

$$\beta(x) = \begin{cases} 1 & , \quad x \in [\frac{4}{3},2] \\ 0 & , \quad otherwise. \end{cases}$$

Now, we prove that the hypotheses (a) and (c) of Corollary 2.5 are satisfied by T and hence T has a fixed point. Let $\alpha(x) \geq 1$ for some $x \in X_\rho$. Then $x \in [-2,\frac{4}{3}]$ and so $Tx \in [\frac{4}{3},2]$. Therefore, $\beta(Tx) \geq 1$. Similarly, if $\beta(x) \geq 1$ then $\alpha(x) \geq 1$. Then T is a cyclic (α,β)-admissible mapping and that the hypotheses (a) and (c) of Corollary 2.5 hold.
Now, for all $x \in [-2,\frac{4}{3}]$ and all $y \in [\frac{4}{3},2]$, we get

$$\begin{aligned} (\alpha(x)\beta(y)+1)^{\psi(\rho(fx-fy))} &= 2^{3\rho(fx-fy)} \\ &= 2^{3\left|\frac{x^2}{3}-\frac{y^2}{3}\right|} \\ &= 2^{|x-y||x+y|} \\ &\leq 2^{2|x-y|} = 2^{3|x-y|-|x-y|} \\ &= 2^{\psi(\rho(x-y))-\phi(\rho(x-y))} \end{aligned}$$

Otherwise, if $\alpha(x)\beta(y) = 0$, we have

$$(\alpha(x)\beta(y)+1)^{\psi(\rho(fx-fy))} = 1 \leq 2^{\psi(\rho(x-y))-\phi(\rho(x-y))}$$

Therefore, Corollary 2.5 implies that T has a fixed point.

Corollary 2.7. *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be a cyclic (α,β)-admissible mapping. Assume that there exists $\ell > 1$ such that*

$$(\psi(\rho(Tx-Ty))+\ell)^{\alpha(x)\beta(y)} \leq \psi(\rho(x-y)) - \phi(\rho(x-y)) + \ell$$

for all $x,y \in X_\rho$ where $\psi \in \Psi$ and $\varphi \in \Phi$. Suppose that the following assertions hold:

(a) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,

(b) T is continuous, or

(c) if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$,

then T has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x,y \in Fix(T)$, then T has a unique fixed point.

Example 2.8. *Let $X_\rho = R^+$, $\rho(x) = |x|$ for all $x \in X_\rho$, and $T : X_\rho \to X_\rho$ by*

$$Tx = \begin{cases} \frac{x^2+x}{4} & , \quad x \in [0,1] \\ 2x & , \quad otherwise. \end{cases}$$

Define $\psi,\phi : [0,+\infty) \to [0,+\infty)$ by $\psi(t) = t, \phi(t) = \frac{t}{4}$ and $\alpha,\beta : X_\rho \to [0,+\infty)$ by

$$\alpha(x) = \beta(x) = \begin{cases} 1 & , \quad x \in [0,1] \\ 0 & , \quad otherwise. \end{cases}$$

Now, we prove that the hypotheses (a) and (c) of Corollary 2.7 are satisfied by T and hence T has a fixed point. Proceeding as in the Example 2.6, we deduce that T is a cyclic (α,β)-admissible mapping and that the hypotheses (a) and (c) of Corollary 2.7 hold.
Now, for all $x \in [0,1]$ and all $y \in [0,1]$, we get

$$\begin{aligned}(\psi(\rho(Tx-Ty))+\ell)^{\alpha(x)\beta(y)} &= |Tx-Ty|+\ell \\ &\leq \frac{1}{4}|x-y|\,|x+y+1|+\ell \\ &\leq \frac{3}{4}|x-y|+\ell \\ &= |x-y|-\frac{1}{4}|x-y|+\ell \\ &= \psi(\rho(x-y))-\phi(\rho(x-y))+\ell.\end{aligned}$$

Otherwise, if $\alpha(x)\beta(y)=0$, we have

$$(\psi(\rho(Tx-Ty))+\ell)^{\alpha(x)\beta(y)} = 1 \leq \psi(\rho(x-y))-\phi(\rho(x-y))+\ell.$$

Therefore, Corollary 2.7 implies that T has a fixed point.

Definition 2.9. *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be mappings. Let the pair T a cyclic $(\alpha,\beta)-$admissible mapping. We say that T is a weak $\alpha-\beta-\psi$- rational contraction if $\alpha(x)\beta(y) \geq 1$ for some $x,y \in X$ implies*

$$\rho(Tx-Ty) \leq M(x,y)-\psi(M(x,y))$$

where $\psi \in \Psi$ and

$$M(x,y) = \max\{\rho(x-y),\rho(x-Tx),\rho(y-Ty),\frac{[1+\rho(x-Tx)]\rho(y-Ty)}{\rho(x-y)+1}\}.$$

Now we are ready to prove our new second result.

Theorem 2.10. *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be a weak $\alpha-\beta-\psi$- rational contraction. Assume that the following assertions hold:*

(a) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,

(b) T is continuous, or

(c) if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$,

then T has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x,y \in Fix(T)$, then T has a unique fixed point.

Proof. Define a sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in N$. Since T is a cyclic (α,β)-admissible mapping and $\alpha(x_0) \geq 1$ then $\beta(x_1) = \beta(Tx_0) \geq 1$ which implies $\alpha(x_2) = \alpha(Tx_1) \geq 1$. By continuing this process, we get $\alpha(x_{2n}) \geq 1$ and $\beta(x_{2n-1}) \geq 1$ for all $n \in N$. Again, since T is a cyclic (α,β)-admissible mapping and $\beta(x_0) \geq 1$, by the similar method, we have $\beta(x_{2n}) \geq 1$ and $\alpha(x_{2n-1}) \geq 1$ for all $n \in N$. That is, $\alpha(x_n) \geq 1$ and $\beta(x_n) \geq 1$ for all $n \in N \cup \{0\}$. Equivalently, $\alpha(x_{n-1})\beta(x_n) \geq 1$ for all $n \in N$. Since T a weak $\alpha-\beta-\psi$ -rational contraction, we get

$$\rho(x_n-x_{n+1}) \leq M(x_{n-1},x_n)-\psi(M(x_{n-1},x_n)) \tag{2.9}$$

where

$$\begin{aligned}M(x_{n-1},x_n) &= \max\{\rho(x_{n-1}-x_n),\rho(x_{n-1}-Tx_{n-1}),\rho(x_n-Tx_n),\frac{[1+\rho(x_{n-1}-Tx_{n-1})]\rho(x_n-Tx_n)}{\rho(x_{n-1}-x_n)+1}\} \\ &= \max\{\rho(x_{n-1}-x_n),\rho(x_n-x_{n+1})\}.\end{aligned}$$

Now, suppose that there exists $n_0 \in N$ such that $\rho(x_{n_0}-x_{n_{0+1}}) > \rho(x_{n_{0-1}}-x_{n_0})$. Therefore $M(x_{n_{0-1}},x_{n_0}) = \rho(x_{n_0}-x_{n_{0+1}})$ and so from (2.9), we get

$$\rho(x_{n_0}-x_{n_0+1}) \leq \rho(x_{n_0}-x_{n_{0+1}})-\psi(\rho(x_{n_0}-x_{n_{0+1}})).$$

This imlies that $\psi(\rho(x_{n_0}-x_{n_0+1})) = 0$, i.e., $\rho(x_{n_0}-x_{n_0+1}) = 0$, which is a contradiction. Hence, $\rho(x_n-x_{n+1}) \leq \rho(x_{n-1}-x_n)$ for all $n \in N$. That is the sequence $\{\rho_n : \rho(x_n-x_{n+1})\}$ is decreasing and so there exists $r \geq 0$ such that $\rho_n \to r$ as $n \to \infty$. Taking the limit as $n \to \infty$ in (2.9), we have

$$r \leq r-\psi(r).$$

This implies that $\psi(r) = 0$. Therefore, the property of ψ implies that $r = 0$. That is

$$\lim_{n\to\infty}\rho(x_{n+1}-x_n) = 0. \tag{2.10}$$

We will show that in this case, $\{x_n\}$ is a Cauchy. Suppose to the contrary. Then there are $\varepsilon > 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, $n(k) > m(k) > k$,

$$\rho(x_{2n(k)} - x_{2m(k)}) \geq \varepsilon \text{ and } \rho(2(x_{2n(k)-1} - x_{2m(k)})) < \varepsilon. \tag{2.11}$$

Now for all $k \in N$, we have

$$\varepsilon \quad \leq \quad \rho(x_{2n(k)} - x_{2m(k)}) \leq \rho(2(x_{2n(k)} - x_{2n(k)-1})) + \rho(2(x_{2n(k)-1} - x_{2m(k)})) < \rho(2(x_{2n(k)} - x_{2n(k)-1})) + \varepsilon.$$

Taking the limit as $k \to \infty$ in above inequality and from (2.10), we have

$$\lim_{k\to\infty} \rho(x_{2n(k)} - x_{2m(k)}) = \varepsilon. \tag{2.12}$$

Since

$$\rho(x_{2n(k)+1} - x_{2m(k)+1}) = \rho(x_{2n(k)+1} - x_{2n(k)} + x_{2n(k)} - x_{2m(k)+1}) \leq \rho(2(x_{2n(k)+1} - x_{2n(k)})) + \rho(2(x_{2n(k)} - x_{2m(k)+1}))$$

and

$$\rho(2(x_{2n(k)} - x_{2m(k)+1})) = \rho(2(x_{2n(k)} - x_{2m(k)} + x_{2m(k)} - x_{2m(k)+1})) \leq \rho(4(x_{2n(k)} - x_{2m(k)}) + \rho(4(x_{2m(k)} - x_{2m(k)+1}))$$

then by taking the limit as $k \to +\infty$ in above inequality and using (2.12) and (2.11), we deduce that

$$\lim_{k\to\infty} \rho(x_{2n(k)} - x_{2m(k)+1}) = \varepsilon. \tag{2.13}$$

Now, by (2.1), we get

$$\rho(x_{2n(k)+1} - x_{2m(k)+1}) \leq M(x_{2n(k)} - x_{2m(k)}) - \varphi(M(x_{2n(k)} - x_{2m(k)})) \tag{2.14}$$

where

$$M(x_{2n(k)} - x_{2m(k)}) = \max\{\rho(x_{2n(k)} - x_{2m(k)}), \rho(x_{2n(k)} - x_{2n(k)+1}), \rho(x_{2m(k)} - x_{2m(k)+1}), \frac{[1+\rho(x_{2n(k)} - x_{2n(k)+1})]\rho(x_{2m(k)} - x_{2m(k)+1})}{\rho(x_{2n(k)} - x_{2m(k)})+1}\}.$$

Letting $k \to \infty$ in (2.14) and using (2.10), (2.12) and (2.13), we get

$$\varepsilon \leq \varepsilon - \psi(\varepsilon).$$

That is $\varepsilon = 0$, which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence. Since X_ρ is complete, then there exists $z \in X_\rho$ such that $x_n \to z$. Suppose that (c) is held. That is, $\alpha(x_{2n})\beta(z) \geq 1$. Since T is a weak $\alpha - \beta - \psi$-rational contradiction, then

$$\rho(x_{2n+1} - Tz) \leq M(x_{2n}, z) - \psi((x_{2n}, z)) \tag{2.15}$$

where

$$M(x_{2n}, z) \quad = \quad \max\{\rho(x_{2n} - z), \rho(x_{2n} - Tz), \rho(z - Tz), \frac{[1 + \rho(x_{2n} - x_{n+1})]\rho(z - Tz)}{\rho(x_{2n} - z) + 1}\}.$$

Taking the limit as $n \to \infty$ in (2.15), we have $z = Tz$. Now, let us to show that T has at most one fixed point. Indeed, if $x,y \in X_\rho$ be two fixed points of T, that is, $Tx = x \neq y = Ty$. $\alpha(x)\beta(y) \geq 1$, it follows that

$$\psi(\rho(x - y)) \leq \psi(M(x, y)) - \phi(M(x, y))$$

where

$$M(x, y) \quad = \quad \max\{\rho(x - y), \rho(x - Tx), \rho(y - Ty), \frac{[1 + \rho(x - Tx)]\rho(y - Ty)}{\rho(x - y) + 1}\}.$$

Then we get

$$\psi(\rho(x - y)) \leq \psi(\rho(x - y)) - \phi(\rho(x - y)).$$

So $\phi(\rho(x - y)) = 0$ and hence, $\rho(x - y) = 0$, taht is, $x = y$.

□

We obtain the following corollaries from Theorem 2.10.

Corollary 2.11. *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be a weak $\alpha - \beta - \psi$- rational contraction such that*

$$\alpha(x)\beta(y)\rho(Tx - Ty) \leq M(x, y) - \psi(M(x, y)),$$

for all $x, y \in X_\rho$ where $\psi \in \Psi$. Suppose that the following assertions hold:

(a) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,

(b) T is continuous, or

(c) if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$,

then T has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Corollary 2.12. *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be a weak $\alpha - \beta - \psi$- rational contraction such that*

$$(\alpha(x)\beta(y)+1)^{\rho(Tx-Ty)} \leq 2^{M(x,y)-\psi(M(x,y))}$$

for all $x, y \in X_\rho$ where $\psi \in \Psi$. Suppose that the following assertions hold:

(a) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,

(b) T is continuous, or

(c) if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$,

then T has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Corollary 2.13. *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be a weak $\alpha - \beta - \psi$- rational contraction. Assume that there exists $\ell > 1$ such that*

$$(\alpha x)(\beta x)+\ell)^{(\alpha x)(\beta x)} \leq M(x,y) - \psi(M(x,y)) + \ell$$

for all $x, y \in X_\rho$ where $\psi \in \Psi$ and $l > 0$. Suppose that the following assertions hold:

(a) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,

(b) T is continuous, or

(c) if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$,

then T has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

3. Application

In this section, firstly we shall apply Corollary 2.3 to show the existence of solution of integral equation. Let φ be a Musielak-Orlicz function on a measurable space $C = ([0,1], \Lambda, \mu)$, where ρ_φ be a modular defined by

$$\rho_\varphi(u) = \int_0^1 \varphi(s, |u(s)|)ds$$

for $\in u \mathscr{L}^\varphi$ and $\alpha_0 > e$ and $c_0 \in [\frac{e}{\alpha_0}, 1)$. Assume that ρ_φ is convex satisfying the Δ_2-condition. Now, we investigate the existence and uniqueness of solution of integral equation:

$$u(t) = e^{-t} f + \int_0^t e^{s-t} \left(\int_0^1 K(\xi, u(s))d\xi \right) ds,$$

where $K : [0,1] \times \mathscr{L}^\varphi \to \mathscr{L}^\varphi$ is a measurable function satisfying:
(1) $\lim\limits_{\lambda \to 0^+} \int_0^1 \varphi(\xi, \lambda \left| (\int_0^1 K(s,u)ds)\xi \right| d\xi = 0$ for any $u \in \mathscr{L}^\varphi$.
(2) $\left| \int_0^1 (K(\xi, u(s)) - K(\xi, v(s)))d\xi \right| \leq k\,|(u-v)(s)|$ for any $u, v \in \mathscr{L}^\varphi$ with $k \in (0,1)$.
(3) We denote by $B = C([0,1], A)$ the space of all ρ-continuous function from $[0,1]$ into A which is a convex, closed, bounded subset of $\mathscr{L}^\varphi$. So, B a closed, bounded, convex subset of $C([0,1], \mathscr{L}^\varphi)$ satisfying the Δ_2-condition.
Let $T : B \to B$ defined by

$$T(u) = \int_0^1 \frac{c_0}{e} K(s,u)ds.$$

(4) $f \in B$.
(5) There exists $u_0 \in B$ such that $\theta(u_0) \geq 0$, $\eta(u_0) \geq 0$ and

$$\begin{aligned} \theta(u) &\geq 0 \text{ for some } u \in B \text{ implies } \eta(Tu) \geq 0, \\ \eta(u) &\geq 0 \text{ for some } u \in B \text{ implies } \theta(Tu) \geq 0. \end{aligned}$$

(6) if $\{u_n\}$ is a sequence in B such that $\theta(u_n) \geq 0$ for all $n \in N \cup \{0\}$ and $u_n \to u$ as $n \to \infty$, then $\theta(u) \geq 0$.
(7) Let $\alpha, \beta : B \to [0, \infty)$ by

$$\alpha(u) = \begin{cases} 1, & \theta(u) \geq 0 \\ 0, & otherwise \end{cases} \quad \text{and} \quad \beta(v) = \begin{cases} 1, & \eta(u) \geq 0 \\ 0, & otherwise. \end{cases}$$

Theorem 3.1. *Under the assumptions (1)-(7), the integral equation has a solution in $C([0,1], \mathscr{L}^\varphi)$.*

Proof. Firstly, we show that T is ρ-Lipschitz. By (1) $\int_0^1 \varphi(\xi,\lambda|Tu(\xi)|\,d\xi \to 0$ as $\lambda \to 0^+$. Hence the definition of $\mathscr{L}^\varphi$, $Tu \in \mathscr{L}^\varphi$ for any $u \in \mathscr{L}^\varphi$.
Let $x, y \in B$,

$$\begin{aligned}\rho_f(Tu-Tv) &= \rho_f(\frac{c_0}{e}(\frac{e}{c_0}(Tu-Tv)))\\ &\leq \frac{c_0}{e}\rho_f(\frac{e}{c_0}(Tu-Tv))\\ &= \frac{c_0}{e}\int_0^1 \varphi(s,\frac{e}{c_0}|(Tu-Tv)(s)|\,ds\\ &= \frac{c_0}{e}\int_0^1 \varphi(s,\frac{e}{c_0}\left|\int_0^1 (K(\xi,u(s))-K(\xi,v(s)))d\xi\right|)ds.\end{aligned}$$

Therefore by (2)

$$\begin{aligned}\rho_f(Tu-Tv) &\leq \frac{c_0}{e}\int_0^1 \varphi(s,k|(u-v)(s)|)ds\\ &= \frac{c_0}{e}\rho_\varphi(k(u-v))\\ &= \frac{c_0}{e}k\rho_\varphi(u-v).\end{aligned}$$

Then, we get T is ρ-Lipschitz (see Theorem 1.3 in [32]). Also define $\psi,\phi : C([0,1],\mathscr{L}^\varphi) \to C([0,1],\mathscr{L}^\varphi)$ by

$$\psi(u) = u, \text{ and } \phi(u) = (1-\frac{c_0}{e}k)u \text{ for } \frac{c_0}{e}k \in (0,1).$$

Consequently, for all $u, v \in B$ we have

$$\alpha(u)\beta(v)\psi(\rho_\varphi(Tu-Tv)) \leq \psi(\rho_\varphi(u-v)) - \phi(\rho_\varphi(u-v)).$$

It shows that all the hypotheses of Corollary 2.3 are satisfied, hence T has a solution $u \in C([0,1],\mathscr{L}^\varphi)$. □

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