Journal of Radiation Research and Applied Sciences 17 (2024) 100788

Contents lists available at ScienceDirect

Journal of Radiation Research and Applied Sciences

journal homepage: www.journals.elsevier.com/journal-of-radiation-research-and-applied-sciences

A new robust ridge parameter estimator having no outlier and ensuring normality for linear regression model

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ARTICLE INFO

Keywords:
Monte Carlo simulation
Multicollinearity
Ridge regression
Search method

ABSTRACT

In order to accurately estimate the regression coefficients in a multiple linear regression model having multicollinearity, ridge regression is a well-liked biased estimation technique used as an alternative to the least squares method. So far, numerous estimators have been proposed for ridge parameter estimation in the ridge regression method. In this study, a robust ridge parameter estimator that performs better than the 366 different ridge parameter estimators proposed in the literature to date has been developed using the "search method". In contrast to the studies in the literature, in addition to the Mean Square Error (MSE) criterion both the estimators and the developed robust ridge parameter estimator have been assessed in terms of outliers and normality. Furthermore, a simulation design with a total of 6050 cases were included in the study which corresponds to parameter values that varied widely in terms of the number of independent variables (p), sample size (n), correlation coefficient between independent variables (ρ), and standard deviation of errors (σ). Significant inter linear associations among the MSE, outlier detection and normality criteria were found. Results from the conducted simulation study revealed that the proposed robust ridge parameter estimator was the most effective one in terms of these three criteria, regardless of the number of regressors, sample size, multicollinearity level, and standard deviation of errors.

1. Introduction

Multiple linear regression is a powerful statistical technique widely used by researchers to examine the linear relationship between a continuous dependent variable and multiple independent variables. The classical Ordinary Least Squares (OLS) approach, used to estimate the coefficients of this model, is an unbiased with minimum variance method under the assumption that covariates are exogenous to the error, or in the case of fixed covariates, if the error has zero mean. This assumption, however, is violated if there is a close-linear relationship between the regressors, leading to the "multicollinearity" that Frisch (1934) first identified.

Isolating the relationship between each independent variable and the dependent variable is one of the primary aims of multiple linear regression analysis. For any fixed coefficient, it is interpreted as the average change in the dependent variable caused by a one-unit change in the independent variable associated with that coefficient when all other independent variables are held constant in a regression model; this is assuming that only the value of that single independent variable can change. However, when independent variables are correlated, a change in any independent variable will result in a change in the value of another independent variable (Frost, 2019). Multicollinearity can lead to problems for each coefficient, including inaccurate estimates, excessive growth of standard deviations, incorrect t-tests, and inaccurate confidence intervals (Saleh, Arashi, & Kibria, 2019). The relationship between the collinearity level and the standard error of each β_i ($i = 0, \cdots, p$) slope coefficient can be easily seen from the following Equation (1),

$$se(\widehat{\beta}_i) = \sqrt{\frac{\sum (y_i - \widehat{y}_i)^2}{\sum (x_i - \bar{x})^2 (1 - R_i^2)(n - p - 1)}}, \tag{1}$$

where y_i is the observation vector, $\widehat{y}_i$ denotes the estimation vector obtained from $\widehat{Y} = X\widehat{\beta}$ in which X is the $n \times (p+1)$ dimensional design matrix containing constant and p observation vectors, n is the sample size and p is the number of regressors. $(1 - R_i^2)$ represents the tolerance value where R_i^2 is the coefficient of determination obtained from the auxiliary model $(e.g., x_{1i} = \beta_0 + \beta_1 x_{2i} + \cdots + \beta_p x_{pi})$ in which x_i is the

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https://doi.org/10.1016/j.jrras.2023.100788
Received 2 November 2023; Received in revised form 28 November 2023; Accepted 4 December 2023
Available online 14 December 2023
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dependent variable and the other independent variables are predictors. Assume that there are two independent variables in a linear regression model, and their correlation is 0.95. In this case, the tolerance value ($1 - 0.95^2$) is 0.098. However, if the correlation between x_1 and x_2 is 0.25, the tolerance value ($1 - 0.25^2$) increases to 0.438. Accordingly, when the sum of squares of x_1 and y is held constant, as the tolerance value decreases, – that is, as the correlation value between x_1 and x_2 increases – so does the standard error value of the regression coefficients (Hoffman, 2022). The more the correlation approaches to one, the more the variances for the $\widehat{\beta}_i$ coefficients tend to go to infinity, making it difficult to draw reliable statistical inferences about the regression coefficients. It is worth emphasizing that multicollinearity does not have a negative effect on the $\widehat{Y}$ prediction values calculated over the multiple linear regression model. Instead, it only has a negative effect on statistical tests, estimations, and interpretations of each coefficient.

Many techniques for dealing with multicollinearity have been developed, including adding more observations to the sample, principal component analysis, factor analysis, and ridge regression (RR).

Ridge regression is a biased estimation technique and was developed as an alternative to OLS, which is unbiased but prone to produce high variance in coefficient estimations. The basic operation in this method is to add a small positive value (k) to the diagonal elements of the $(p+1)\times(p+1)$ dimensional $(X'X)$ matrix used in OLS coefficient estimation. This approach prevents the overinflating of regression coefficient variances, enhancing the reliability of statistical inferences regarding the coefficients.

Many estimators for estimating the ridge parameter k have been proposed by researchers to date. When evaluating the performances of estimators, Mean Square Error (MSE) criterion has been the only commonly used criterion so far and may be presented as follows:

$$MSE = \frac{\sum_{i=0}^{p} (\beta_i - \widehat{\beta}_i)^2}{p+1} \tag{2}$$

In virtually no study or analysis in this field has a tool been presented taking into account the range of the ridge parameter estimators to determine possible outlier (anomalies) or skewness in the distribution of ridge parameter estimators. From this perspective, Mermi, Göktaş, and Akkuş (2021) evaluated the performance of 80 different ridge parameter estimators proposed in the literature, using the MSE criterion as well as the criteria for the calculated values of ridge parameter estimators to be in the range of 0–10 and assuring normality of ridge parameter estimators. On the other hand, it has been determined that the performances of all ridge parameter estimators proposed in the literature are considered in a very narrow scope in terms of simulation parameters related to the number of regressors (p), sample size (n), correlation level among regressors (ρ) and standard deviation of errors (σ). This highlights the lack of a robust ridge parameter estimator that can be applied in practice and perform well in case of real-life data with a wide range of p, n, ρ, and σ values. In order to fill this gap in the literature, Göktaş, Akkuş, and Kuvat (2021) developed the "search method" and proposed a robust ridge parameter estimator. However, this proposed estimator was obtained by linear combinations of three estimators, which were determined to be the most efficient among only 37 different ridge parameter estimators whose performances were considered by Göktaş and Sevinç (2016) in terms of MSE criterion. In the majority of the 324 cases designed with different values of p, n, and ρ in a Monte Carlo simulation analysis, the robust estimator developed was found to be more efficient than the three estimators in terms of MSE.

This paper primarily aims to develop a robust ridge parameter estimator that outperforms the other 366 estimators in most of the 6050 total cases considered in a wide range of p, n, ρ, and σ, by integrating a total of 366 different ridge parameter estimators proposed in the literature to date into the search method. The reason why we call the developed ridge parameter estimator “robust” is that its use is free of sample size, collinearity level, number of regressors, and error variance. In addition to the MSE criterion, two additional criteria proposed by Mermi et al. (2021) have been improved to assess the performance of these estimators. These updated criteria include outlier detection within calculated ridge parameter estimators based on the "median absolute deviation" method and normality assessment utilizing skewness and kurtosis coefficients. Thereby, out of the 366 ridge parameter estimators proposed in the literature, an estimator that is developed to be robust performs better in terms of these three criteria in the majority of the 6050 cases. In this regard, the proposed robust ridge parameter estimator will resolve the confusion prevailing in the literature over which ridge parameter estimator to use in any case of p, n, ρ, and σ.

Finally, a Spearman’s rank correlation analysis is conducted to determine the intercorrelations between the three different criteria considered for the ridge parameter estimators.

2. Literature review

In ridge regression method, it is crucial to accurately estimate the ridge parameter in accordance with p, n, ρ, and σ parameter values that vary owing to the structure of the data in practice. This section reviews studies on ridge parameter estimation from the development of the ordinary ridge regression model to the present, mostly using the MSE criterion for performance evaluation.

Hoerl and Kennard (1970), who proposed the ridge regression technique, presented an estimator for estimating ridge parameter in their first study. Furthermore, using a graphical technique called "ridge trace," the change in total variance and bias values of regression coefficients, as well as the change in MSE value derived from ridge regression, in response to k values varying in the 0–1 range can be observed. Thus, it has been illustrated that an optimal ridge parameter value with a reasonable variance and bias can be obtained. In another subsequent study, Hoerl, Kennard, and Baldwin (1975) proposed a new ridge parameter estimator, which proved to be superior to OLS through simulation. Later, Lawless and Wang (1976) compared the performance of some proposed estimators for the generalized ridge regression method, and Hoerl et al.’s (1975) estimator, together with a new ridge parameter estimator they proposed. The simulation analysis demonstrated that the proposed estimator, together with the estimator by Hoerl et al. (1975), surpassed the others in terms of two widely used MSE criteria. In further studies, Kibria (2003) proposed three new ridge parameter estimators, examining their performances with simulated and real-world data with the estimators proposed by Hoerl and Kennard (1970), Hoerl et al. (1975) and Lawless and Wang (1976), as well as OLS. It was reported that those new estimators are superior to others under certain conditions. Alkhamisi and Shukur (2007) presented six new ridge parameter estimators, including the estimators proposed, by applying the median and arithmetic mean techniques used by Kibria (2003). It has been revealed by the Monte Carlo simulation study that at least one of the proposed estimators outperforms the OLS and the estimator proposed by Hoerl and Kennard (1970), both in terms of the MSE and proportion of replications criterion for which the OLS estimator produces an MSE smaller than the MSE of others. Dorugade and Kashid (2010) proposed a new ridge parameter estimator based on Hoerl and Kennard’s (1970) estimator and the variance inflation factor of regressors. They have conducted a simulation study which comprised of the estimators proposed by Hoerl and Kennard (1970), Lawless and Wang (1976), Nomura (1988) and Khalaf and Shukur (2005), showing that the proposed estimator is more efficient than the others. Muniz, Kibria, Mansson, and Shukur (2012) proposed five new ridge parameter estimators by modifying Khalaf and Shukur’s (2005) estimator using the square root transformation method developed by Alkhamisi and Shukur (2008). The efficiency of 19 different estimators, including the proposed estimators, was compared through Monte Carlo simulation analysis. It was stated that the proposed estimators perform better than some others. Khalaf and Iguernane (2014) proposed four new ridge parameter

estimators consisting of a kind of function as well as the arithmetic, geometric, and harmonic means of the ridge parameter estimators proposed by Hoerl and Kennard (1970) and Hoerl et al. (1975). The simulation study findings demonstrated that new estimators are more effective than those proposed by Hoerl and Kennard (1970) and Hoerl et al. (1975) under certain conditions.

When it comes to current studies, Dorugade (2016) determined the ridge parameter estimator as the standard deviation of the OLS-estimated regression model, in contrast to previous studies. Simulation-based performance comparisons were carried out with some ridge parameter estimators used in the literature for different types of models contaminated with outliers or errors violating normality. The proposed ridge parameter estimator has proven to be more efficient than the OLS estimator, Hoerl and Kennard's (1970) estimator, the jackknifed modified ridge estimator, and the jackknifed ridge M-estimator. Kibria and Banik (2016) presented five different ridge parameter estimators derived from the arithmetic mean, median, maximum value, geometric mean, and harmonic mean of some previously proposed estimators. Through a simulation study and two real-world data sets, the proposed estimators were compared with 28 different estimators. Their simulation results proved that beside their proposed estimators, some other estimators also performed better than the popular estimators in the study. Lukman, Ayinde, and Ajiboye (2017) applied various transformation techniques used in Lukman and Ayinde's (2017) study to the ridge parameter estimators proposed by Dorugade (2014a) and Karaibrahimoğlu, Asar, and Genç (2016). The performance of the new ridge parameter estimators obtained in this way was compared with some popular estimators in the literature via Monte Carlo simulation. The comparisons indicated that the new ridge parameter estimators obtained based on the estimator proposed by Karaibrahimoğlu et al. (2016) have lower MSE values than the others in general. Lukman and Olatunji (2018) proposed a new estimator consisting of a modification of the ridge parameter estimator proposed by Dorugade (2016) outperforming OLS and the estimators proposed by Hoerl and Kennard (1970), Hoerl et al. (1975) and Dorugade (2016). Lukman, Oluyemi, Akanbi, and Clement (2018) compared the performance of several new ridge parameter estimators obtained using the transformation techniques of Alkhamisi, Khalaf, and Shukur (2006) and Lukman and Ayinde (2017), through their relative efficiency values relative to the OLS estimator. The Monte Carlo simulation analysis revealed that the performance of ridge parameter estimators varies with sample size, degree of multicollinearity, and error variances. Further, it was concluded that under certain conditions, the estimators obtained by some transformation techniques are more effective than others. Bhat and Vidya (2020) took a different approach, defining the first and second largest eigenvalues of the $(X'X)$ matrix as weights. Using these weights, two new ridge parameter estimators based on the weighted geometric and harmonic means of the estimators proposed by Hoerl et al. (1975) and Dorugade and Kashid (2010) were proposed. It was found that the performance of these proposed estimators is more satisfactory than other estimators discussed in the study and provides more stable estimations. Qasim, Mansson, and Kibria (2021) presented a new ridge regression estimation method for the beta regression model to address the instability of the classic maximum likelihood (ML) method in the presence of multicollinearity. In this method, the performance of five new estimators proposed in their study and several popular estimators proposed for the ordinary ridge regression method used for ridge parameter estimation were evaluated via Monte Carlo simulation and real data analysis. They demonstrated that the proposed estimators outperform some others, including the ML approach, in terms of both MSE and median square error criteria. Recently, Kibria (2022) considered more than 100 estimators for estimating ridge or shrinkage parameter k for both linear and non-linear regression models.

Based on those aforementioned major publications with respect to ridge regression analysis, it seems that in most studies the common feature of these estimators is that their performance was mostly evaluated based on only MSE criterion, within a narrow scope in terms of number of regressors, sample size, collinearity level, and errors' variance. This highlights the lack of a robust ridge parameter estimator that can be applied in practice and perform well with a wide range of p, n, ρ, and σ values to fill the gap.

3. Methodology of ridge regression

Let the basic Multivariate Linear Regression (MLR) model be given as;

$$y = X\beta + u, \tag{3}$$

where y is an $n \times 1$ vector of the dependent variable, X is an $n \times p$ design matrix with full rank, β is a vector of unknown parameters with $p \times 1$ dimension and u is a $n \times 1$ dimensional vector of residuals which are independent and identically distributed with zero mean and the σ^2 variance.

The OLS estimator of β coefficients is given as follows:

$$\widehat{\beta}_{OLS} = (X^T X)^{-1} X^T y. \tag{4}$$

The covariance matrix of the OLS estimator is given by

$$Cov(\widehat{\beta}_{OLS}) = \sigma^2 (X^T X)^{-1}. \tag{5}$$

One of the optimality properties of the OLS estimator is unbiasedness, given by Equation (6).

$$E(\widehat{\beta}_{OLS}) = E\left((X^T X)^{-1} X^T y\right) = (X^T X)^{-1} X^T X\beta = \beta. \tag{6}$$

Hoerl and Kennard (1970) proposed ridge regression technique in case of multicollinearity to prevent overinflating the variances of the coefficients estimated by OLS in Equation (5) and the drawing of incorrect statistical inferences.

The objective function that is minimized to produce the ridge regression estimator is as follows:

$$(y - X\beta)^T (y - X\beta) + k\beta^T \beta, \tag{7}$$

where $\beta_{\text{penalized}} = \sum_j \beta_j^2 = \beta^T \beta$ indicates the penalty term applied to the OLS estimator. The ridge regression estimation method obtained as a result of solving the expression given in Equation (7) by taking the first derivative with respect to β and setting it equal to zero is given in Equation (8).

$$\widehat{\beta}_{RR} = \left(X^T X + kI_p\right)^{-1} X^T y, k \geq 0 \tag{8}$$

As seen in Equation (8), the ridge regression approach involves adding a small positive number (k) to the diagonal elements of the $(X'X)$ matrix. This prevents the variance of the regression coefficients (as given by Equation (5)) from overinflating due to multicollinearity.

The penalized OLS estimator is biased, as shown in Equation (9), in contrast to the estimator in Equation (6),

$$E(\widehat{\beta}_{RR}) = E\left(\left(X^T X + kI_p\right)^{-1} X^T y\right) = \left(X^T X + kI_p\right)^{-1} X^T X\beta. \tag{9}$$

since the matrices $(X^T X + kI_p)^{-1}$ and $X^T X$ can not be canceled, unless $k = 0$.

The covariance matrix for the ridge regression estimator can be obtained using the following equation,

$$Cov(\widehat{\beta}_{RR}) = \sigma^2 \left(X^T X + kI_p\right)^{-1} X^T X \left(X^T X + kI_p\right)^{-1}. \tag{10}$$

Since $|Cov(\widehat{\beta}_{OLS})| > |Cov(\widehat{\beta}_{RR})|$, then $|Cov(\widehat{\beta}_{OLS})| - |Cov(\widehat{\beta}_{RR})|$ is positive for $k > 0$. In other words, ridge regression generates estimations with lower variances than OLS, as the covariance matrix obtained by ridge regression is smaller than that of OLS. This is expressed by the following equation (Fahrmeir, Kneib, Lang, & Marx, 2021).

$$Var(\widehat{\beta}_{j,RR}) < Var(\widehat{\beta}_{j,OLS}), j = 1, \cdots, p. \quad (11)$$

For the convenience of calculation, the usual form of the multiple linear regression model shown in Equation (3) can be expressed as $y = T\alpha + u$ in the canonical form, where $T = XQ$, $T^TT = Q^TX^TXQ = W = diag(\lambda_1, \cdots, \lambda_p)$ and $\alpha = Q^T\beta$, λ_j's are the j^{th} eigenvalue of the matrix (X^TX) and Q is $p \times p$ orthogonal eigenvectors matrix corresponding to (X^TX). Hence, the OLS estimator of α is obtained by $\widehat{\alpha}_{OLS} = (T^TT)^{-1}T^Ty = W^{-1}T^Ty$. The variance-covariance matrix of $\widehat{\alpha}_{OLS}$ is given by $var(\widehat{\alpha}_{OLS}) = \sigma^2W^{-1}$.

On the other hand, the ridge estimator of α is given by $\widehat{\alpha}_{RR} = (W + kI_p)^{-1}T^Ty$ and the variance turns into the $var(\widehat{\alpha}_{RR}) = \sigma^2(W + kI_p)^{-1}W(W + kI_p)^{-1}$ form. Accordingly, the total variance of $\widehat{\alpha}_{RR}$ coefficients can be expressed as $tr(var(\widehat{\alpha}_{RR})) = \sigma^2\sum_{i=1}^{p}\frac{\lambda_i}{(\lambda_i+k)^2}$.

The mean square error of the $\widehat{\alpha}_{RR}$ coefficients can be expressed as the sum of the variances and the squared sum of the bias values of the $\widehat{\alpha}_{RR}$, as follows:

$$\begin{aligned} MSE(\widehat{\alpha}_{RR}(k)) &= \sigma^2\sum_{i=1}^{p}\frac{\lambda_i}{(\lambda_i+k)^2} + k^2\sum_{i=1}^{p}\left(\frac{\widehat{\alpha}_i}{\lambda_i+k}\right)^2, \\ &= Var(\widehat{\alpha}_{RR}(k)) + (bias(\widehat{\alpha}_{RR}(k))^2. \end{aligned} \quad (12)$$

It is obvious that as k increases, the total variances associated with $\widehat{\alpha}_{RR}$ decreases and the bias value increases. When $k = 0$, the formula equals the MSE obtained from OLS. As a result, a positive value of k to be estimated will slightly raise the bias of $\widehat{\alpha}_{RR}$ coefficients, but significantly reduce their variances (Saleh et al., 2019).

3.1. Search method algorithm

Göktaş et al. (2021) developed a robust ridge parameter estimator with their proposed "search method". The starting point of this method is the fact that the parameter value k is unique, yielding the minimum MSE value in any case to be considered in terms of p, n, ρ, and σ. From this point of view, the proposed search method basically consists of the following algorithmic steps:

1. A dataset is derived from the simulation design composed of parameters p, n, ρ, and σ. To estimate the parameters of the ridge regression model, Equation (8) is applied such that the value of k is ranging from 0 to 500 with an increment of 0.01. For each k, the corresponding MSE is calculated as defined in Equation (2). The value of k that corresponds to the smallest calculated MSE out of the 50000 computed MSE values is considered to be robust.
2. Use the same dataset generated in step 1 to calculate the s different number of ridge parameter estimators.
3. Repeat Step 1 and Step 2 for 30 times.
4. Repeat the process carried out in first three steps for different values of p, n, ρ, and σ variables.
5. Remove the rows where k is zero from dataset.
6. Fit the multiple linear regression model with all s number of ridge parameter estimators considered in the study, along with the independent variables p, n, ρ, and σ with respect to the dependent variable, which is the natural logarithm of $ln(k)$ determined.

$$ln(k) = \theta_1k_1' + \theta_2k_2' + \cdots + \theta_sk_s' + \theta_{s+1}p + \theta_{s+2}n + \theta_{s+3}\rho + \theta_{s+4}\sigma + \varepsilon. \quad (13)$$

7. Using OLS, obtain the model with $l\,(l < s)$ number of independent variables determined to be the most significant to provide the estimator for k_{robust} parameter as in Equation (14).

$$k_{robust} = exp(\theta_1k_1' + \theta_2k_2' + \cdots + \theta_lk_l' + \theta_{l+1}p + \theta_{l+2}n + \theta_{l+3}\rho + \theta_{l+4}\sigma + \varepsilon). \quad (14)$$

The search method outlined above quarantees that the developed k_{robust} outperforms all other ridge parameter estimators included in the method in terms of MSE, outlier detection, and normality criteria in most cases covering p, n, ρ, and σ extensively.

4. Proposed 366 different ridge parameter estimators so far

In this section, we present a total of 366 different ridge parameter estimators proposed or discussed by researchers from Hoerl and Kennard (1970) to date. The majority of these estimators were proposed for linear regression models. Estimators proposed in the literature for models such as logistic regression, beta regression, and Poisson regression were adapted to ORR and included in this study. It should also be stated that the ridge parameter estimators proposed in prior studies have been cited based on their proposed study. Accordingly, a total of 366 different ridge parameter estimators are listed in the Appendix.

5. Simulation study

In this section of the study, two different Monte Carlo simulation designs were conducted to evaluate the performance of ridge parameter estimators in regards to MSE, outlier detection, normality criteria, and for the search method stage to be used to develop a robust ridge parameter estimator. Following Gibbons (1981) and Kibria (2003), the data generation process of the independent variables was carried out in MATLAB 2022, using the following universal formulation,

$$x_{ij} = \left(1 - \rho^2\right)^{1/2}z_{ij} + \rho z_{i,p+1}, i = 1, 2, \cdots, n \text{ and } \quad j = 1, 2, \ldots, p. \quad (15)$$

where z_{ij}'s represent the independent pseudo-random variable derived from the standard normal distribution, and ρ denotes the degree of correlation between regressors. Then, the dependent variable y was obtained by plugging into Equation (16):

$$y_i = \beta_0 + \beta_1x_{i1} + \beta_2x_{i2} + \cdots + \beta_px_{ip} + \varepsilon_i, i = 1, 2, \cdots, n. \quad (16)$$

where ε_i are pseudo-random numbers distributed normally with zero mean and σ^2 variance, $\varepsilon_i \sim N(0, \sigma^2)$.

Many factors can impact the performance of ridge parameter estimators for different evaluation criteria. In this study, 6050 different cases were evaluated by considering 10 different values for regressors ($p = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$), 11 different correlation levels ($\rho = 0.1$, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95, 0.99), 11 different sample sizes ($n = 20, 25, 30, 50, 80, 100, 120, 150, 250, 500, 1000$) and 5 different error standard deviation values ($\sigma = 1, 2, 3, 4, 5$). The predictor variables were generated using p, ρ, n, and σ. The experiment was iterated N = 50000 times, and the estimation values produced by the robust estimator and 366 "k_i" (i = 1, 2, ⋯,366) ridge parameter estimators were computed at each iteration. The $\widehat{\beta}_{RR}$ coefficients corresponding to these k_i values were then estimated using the ridge regression approach presented in Equation (8). In the data generation process, the actual coefficients (β) for convenience were set to "1″ for all regressors, and the error between the actual and estimation values ($\widehat{\beta}$) was calculated for each case using the MSE criterion given by Equation (2).

In order to assess the performance of estimators based on the MSE criterion for N = 50000 iterations, the average MSE (AMSE) was calculated using Equation (17):

$$AMSE = \frac{1}{(p+1) \times 50000}\sum_{i=0}^{p}(\widehat{\beta}_i - \beta)'(\widehat{\beta}_i - \beta). \quad (17)$$

In order to prove the success of the proposed k_{robust} ridge parameter estimator, the ratio of having the minimum AMSE value within 6050 different cases was obtained initially. This was accomplished by counting the number of minimum AMSE values among 366 estimators and dividing by the number of 6050 cases.

The AMSE statistic may be impacted by outlier MSE values generated by the ridge parameter estimators, leading to substantial deviations

from the center of the data. In order to disregard the effect of these outliers, the median MSE values were also computed in addition to the average MSE values calculated for each case with N = 50000 replications. The ratio of minimum median MSE for k_{robust} estimator among 366 estimators was then determined based on the computed median MSE values, thus eliminating the impact of MSE outlier values.

However, in some cases, despite having the minimum average MSE value among the rankings based on average MSE value, the k_{robust} estimator may not have the minimum median MSE value in the rankings based on median MSE. This corresponds to cases in which the estimators do not fairly generate outlier MSE values. In such cases, k_{robust} estimator was considered to have the minimum average MSE value, taking into account the rankings based on average MSE instead of the rankings based on median MSE.

Another factor to consider is that there may be cases where k_{robust} does not have a minimum AMSE value, but where there is no statistically significant difference between the estimators with smaller AMSE values. To determine such cases, ANOVA tests were conducted to check if there was a significant difference between the AMSE value of k_{robust} and the estimators presenting smaller AMSE values. However, since there were N = 50000 iterations for each case, the ANOVA test statistic value may have been high due to the large sample size, leading to incorrect rejection of the null hypothesis of no significant difference between the estimators. See Royall (1986) and Armstrong (2019) for more information on the negative effects of large samples on statistical significance tests. To address this, ANOVA tests were performed by randomly selecting 500 samples from k_{robust} and the estimators with smaller AMSE values in cases where k_{robust} did not have a minimum AMSE value. When there was no statistically significant difference ($p > 0.05$), it was statistically accepted that the difference between the k_{robust} estimator and the ridge parameter estimator with minimum AMSE value was not significant, and k_{robust} was considered to have the minimum AMSE value.

The ratio of outliers generated by the ridge parameter estimators in all cases is evaluated as a secondary performance criterion. Outliers common in statistical analysis can significantly impact the accuracy of estimations and inferences regarding the model being analyzed. In this respect, ridge parameter estimators that generate significant outliers may lead ridge regression coefficient estimations to deviate considerably from actual parameter values, resulting in high MSE values. Furthermore, it may result in a skewed distribution of estimators' values. Therefore, the problem of outlier generation in ridge parameter estimators should not be ignored. Many outlier detection methods exist, depending on whether the data is univariate or multivariate, and whether it is normally distributed. The outlier detection method based on MAD (Median Absolute Deviation: MAD) under the assumption of univariate and normality is one of these commonly used methods. This median-based method is more effective than the mean and standard-deviation based z-scores method. This is because the median is less affected by outliers in the data than the mean and standard deviation. Additionally, it does not suffer from the drawbacks of the z-scores method, such as the necessity for normality (including outliers) and poor performance in small sample sizes.

In this study, the following inequality range is utilized for the diagnosis of outliers,

$$M - 3 \times MAD < x_i < M + 3 \times MAD, i = 1, \cdots, n, \tag{18}$$

where $MAD = c \times M(|x_i - M(x_i)|)$, x_i represents n observations points, M indicates the median of series, and c is a correction parameter based on the normality assumption of data. This constant is accepted as 1.4826 (Leys, Ley, Klein, Bernard, & Licata, 2013). Accordingly, the number and ratio of outliers were calculated using the MAD-based method given in (18) for N = 50000 values produced by 366 estimators and k_{robust} in each of 6050 cases. The overall outlier generation ratios were computed by taking the average of the ratios calculated in each case.

Finally, since the values derived from the ridge parameter estimators are expected to satisfy normality, the normality criterion was also included along with MSE and outlier detection criteria. The normality of the ridge parameter estimators was determined by means of absolute skewness and kurtosis measures of derived values based on the studies of Kim (2013) and West, Finch, and Curran (1995). Hence, an estimator was considered to provide normality in cases where the absolute skewness value was less than 2 and the absolute kurtosis value was less than 7. This process for each estimator was performed for 6050 different simulation designs, and the ratio of estimators with normal distribution was obtained.

5.1. Development of robust ridge parameter estimator via search method

In this study, algorithmic steps related to the search method introduced in Section 3.1 were applied to develop the robust ridge parameter estimator. First of all, our preliminary research revealed that in the search method stage the k parameter values that give minimum MSE amounts at very small values of σ are zero and grow excessively at very large values, causing outliers. It should also be noted that the standard deviation value (σ) of errors is not expected to be extremely large or extremely small in practice. Therefore, five standard deviation values of errors (σ = 1, 2, 3, 4, 5) were included in the study as a simulation parameter. So, 2420 designs out of a total of 6050 were included in the search method, considering four different numbers of regressors (p = 3, 5, 7, 11), eleven different correlation levels (ρ = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95, 0.99) and eleven different sample sizes (n = 20, 25, 30, 50, 80, 100, 120, 150, 250, 500, 1000), and data was generated using Equation (15).

Following the first six algorithmic steps, the logarithmic linear regression model corresponding to Equation (13) includes $s = 366$ different ridge parameter estimators. Among these variables, a total of $l = 10$ estimators determined to be the most significant. The final form of the robust ridge parameter estimator, which includes 10 estimators along with the p, n, ρ, and σ independent variables is provided in the following equation.

$$k_{robust} = \exp\left(-0.123k_1' - 0.360k_2' - 184.870k_3' + 0.575k_4' + 0.135k_5' - 1.637k_6' + 1.237k_7' + 184.900k_8' - 0.059k_9' - 0.004k_{10}' + 0.020p + 0.002n + 0.503\sigma\right). \tag{19}$$

The parameters from k_1' to k_{10}' in Equation (19) correspond to ridge parameter estimators k_{15}, k_{30}, k_{41}, k_{62}, k_{181}, k_{182}, k_{218}, k_{305}, k_{313}, and k_{328} in the study, respectively. The ANOVA test for the model obtained in Equation (19), as well as student-t test results for each independent variable, are significant at the 0.05 significance level ($p < 0.05$). The model has an R^2 value of 98.28%, meaning the independent variables k_{15}, k_{30}, k_{41}, k_{62}, k_{181}, k_{182}, k_{218}, k_{305}, k_{313}, k_{328}, p, n, ρ, and σ succeed in explaining the majority of variance (98.28%) in the k_{robust} estimator.

The independent variables k_{15}, k_{30}, k_{41}, k_{62}, k_{181}, k_{182}, k_{218}, k_{305}, k_{313} and k_{328} that were found to be most significant in the creation of the robust estimator model can be given again as follows.

$$\boldsymbol{k_{15}} = \frac{\widehat{\sigma}^2}{\widehat{\beta}_{max}^2} + \frac{1}{\lambda_{max}} \boldsymbol{k_{30}} = \prod_{i=0}^{p} \left(\frac{\lambda_{max}}{(n-p)\widehat{\sigma}^2 + \lambda_{max}\widehat{a}_i^2} \right)^{\frac{1}{p}} \boldsymbol{k_{41}} = \frac{p\widehat{\sigma}^2}{\widehat{\beta}'\widehat{\beta}} + \frac{1}{\lambda_{max}\widehat{a}'\widehat{a}}$$

$$k_{62}=\left(\prod_{i=1}^{p}\sqrt{\frac{(\lambda_{max}\widehat{\sigma}^2)}{((n-p)\widehat{\sigma}^2+\lambda_{max}\widehat{a}_i^2)}}\right)^{\frac{1}{p}}\quad k_{181}=\frac{1}{p}\sum_{i=1}^{p}1\Big/\sqrt{\frac{\widehat{\sigma}^2}{\lambda_i\widehat{a}_i^2}}\quad k_{182}=p\Big/\sum_{i=1}^{p}\sqrt{\frac{\lambda_i\widehat{a}_i^2}{\widehat{\sigma}^2}}$$

$$k_{218}=\sqrt{p\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+[\max(\lambda_i)]\widehat{a}_i^2}}\quad k_{305}=\frac{p\widehat{\sigma}^2}{\widehat{\alpha}'\widehat{\alpha}}+\frac{1}{n\lambda_{max}}\quad k_{313}=min\left(\sqrt{\frac{\lambda_{min}\sum_{i=1}^{p}\widehat{\alpha}_i^2}{\widehat{\sigma}^2}}\right)$$

$$k_{328}=\left(\prod_{i=1}^{p}1\Big/\sqrt{\frac{(\lambda_{max}\widehat{\sigma}^2)}{((n-p)\widehat{\sigma}^2+\lambda_{max}\widehat{a}_i^2)}}\right)^{1/p}\frac{\lambda_{max}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$$

5.2. Performance of k_{robust} and 366 ridge parameter estimators in terms of MSE criteria

According to the number of independent variables, Table 1 presents the percentages of cases where the k_{robust} estimator has the minimum MSE values based on 1) average MSE, 2) median MSE, 3) average MSE or median MSE, and 4) including ANOVA tests, among a total of 6050 cases.

According to Table 1, the percentage of the k_{robust} estimator having the minimum AMSE value, including the effect of outlier MSE values, is 54.79% among 6050 cases. Independently from Tables 1 and it was determined that estimators with the minimum AMSE value after the k_{robust} estimator among all cases were k_{149} second with 3.29%, k_{219} third with 2.89%, k_{211} fourth with 2.21%, and k_{209} fifth with 2.18%. These percentages show that the k_{robust} estimator obviously surpasses other estimators in terms of having a minimum MSE value criterion for all cases. The impact of the outlier MSE values produced by ridge parameter estimators is eliminated using median MSE-based rankings, resulting in a reduction of minimum median MSE value for the k_{robust} estimator to 51.22%. When considering both the average MSE and median MSE criteria, the percentage of k_{robust} estimator having at least one of the minimum AMSE and minimum median MSE values increases to 57.54%.

Lastly, it has been investigated via ANOVA tests whether there is a significant difference between k_{robust} and the estimators up to the estimator having the minimum AMSE value in cases where k_{robust} is larger than the minimum AMSE value. In cases where there was no significant difference, k_{robust} was treated as the estimator having minimum AMSE value. As a result, the percentage of k_{robust} estimator having the minimum AMSE value among all cases has been increased to 96.28%. It can therefore be concluded that the newly developed k_{robust} estimator has the minimum AMSE value for N = 50000 iterations in 96.28% of the 6050 different cases discussed in the paper.

This process was not only applied for k_{robust} but also for ridge parameter estimators that had the highest percentage of the lowest AMSE values after the k_{robust} (k_{149}, k_{219}, k_{211}, and k_{209}). Using this procedure, the percentage with minimum AMSE value for those opponent ridge parameters has been significantly increased. The percentage with minimum AMSE value was increased to 22.81% for k_{149}, 28.60% for k_{219}, 28.91% for k_{211}, and 25.11% for k_{209}. When compared with the k_{robust} estimator, which has the minimum AMSE value in 96.28% of all cases, it was seen that the estimators following k_{robust} did not exceed 30%. This reveals that the k_{robust} estimator developed outperforms ridge parameter estimators k_{149}, k_{219}, k_{211} and k_{209}, which have the best performance following k_{robust} in terms of the lowest AMSE value. Moreover, it has been determined that these estimators do not perform adequately when evaluated against criteria to be examined later in this study.

The following technique was developed to determine the performance ranking of the k_{robust} estimator we developed and 366 ridge parameter estimators in terms of all cases: First, rank numbers were assigned to the calculated AMSE values of the k_{robust} and 366 ridge parameter estimators for each of 6050 cases. The "sum of rank numbers" values were then calculated by adding up the rank numbers of each estimator for all cases. At this point, the fact that an estimator has the smallest sum of rank numbers can be evaluated, as that estimator generally has the smallest rank numbers in all the cases discussed. For this reason, the estimators with the smallest sum of rank numbers were determined to be the most efficient among all estimators in terms of the MSE criterion. Accordingly, the minimum sum of rank numbers that an estimator can get is 6050, the total number of cases, assuming that this estimator is first in all cases. On the other hand, if an estimator is considered to be in the last rank (367th) of all cases, the maximum sum of rank numbers will be equal to the product of the total number of cases and the total number of estimators (367x6050 = 2220350).

Table 2 lists the top 50 most efficient ridge parameter estimators in terms of MSE criterion based on the sum of rank numbers. Additionally, for these estimators, the average percentages of outliers in all cases are presented using the outlier detection method based on the MAD given in Equation (18). Finally, the percentages of normality cases, determined based on absolute skewness and kurtosis values of the ridge parameter estimators, within a total of 6050 cases were calculated. The results of other ridge parameter estimators, that are not included in this table, can be shared with researchers who are interested if the authors are contacted.

The results in Table 2 indicate that k_{robust} is the most successful estimator in terms of MSE criterion based on the sum of rank numbers. Considering the proposed k_{robust} estimator, it can be stated that it performs quite well in terms of both outlier generation (0.5%) and normality (99.82%) criteria. Despite the fact that the outlier percentage of the second-placed estimator k_{27} was 3.65%, the normality percentage remained at 62.36%. It is also seen that the third-placed estimator k_{44} has a high outlier percentage (10.31%) and a low normality percentage (16.66%).

On the other hand, the ridge parameter estimators with minimum AMSE value following k_{robust}, were determined to be k_{149}, k_{219}, k_{211}, and k_{209}. Regardless of Table 2 (except k_{211}), the percentages of outlier generation and normality for all cases of these estimators, respectively, were determined as 3.17% and 64.48% for k_{149}, 3.80%, and 66.94% for k_{219}, 4.17% and 70.64% for k_{211}, 8.08%, and 35.55% for k_{209}. Although they have the minimum AMSE value after k_{robust}, these estimators cannot be considered effective since they fail to meet at least one of the outlier generation and normality criteria.

In conclusion, it is seen that commonly encountered ridge parameter estimators in literature were recommended by evaluating based solely on the MSE criterion. Although a ridge parameter estimator is effective

Table 1
Percentages of the k_{robust} estimator having some certain minimum MSE values and including ANOVA tests for each number of regressors.

	Percentage of having the minimum MSE according to average MSE (%)	Percentage of having the minimum MSE according to median MSE (%)	Percentage of having the minimum MSE according to average MSE or median MSE (%)	Percentage of having minimum MSE including ANOVA tests (%)
$p = 2$	59.34	61.16	66.12	97.69
$p = 3$	62.81	60.83	68.10	98.35
$p = 4$	62.81	57.36	65.79	98.84
$p = 5$	60.50	55.54	62.31	98.35
$p = 6$	59.17	55.54	61.32	98.51
$p = 7$	55.37	50.91	57.52	97.19
$p = 8$	51.07	46.94	53.39	96.03
$p = 9$	48.76	44.30	49.92	94.21
$p = 10$	45.29	41.16	46.61	92.56
$p = 11$	42.81	38.51	44.30	91.07
All cases	**54.79**	**51.22**	**57.54**	**96.28**

for the MSE criterion, it cannot be considered useable unless the outlier rate is low and the normality rate is high. If an estimator generates outliers that exceed acceptable limits and does not meet normality, it may lead to incorrect calculations and misinterpretations of the estimated parameters in the ridge regression model.

5.3. Performances of k_{robust} and 366 ridge parameter estimators in terms of outlier generation and normality criteria

In order to evaluate the performance of ridge parameter estimators based on the outlier generation criterion, the first 20 estimators with the lowest and highest outlier generation percentages are given in Table 3.

Independently and in addition to Tables 3 and it was found that 58 out of 366 ridge parameter estimators had outlier generation percentages above 5%, while 109 were below 1%. It should be noted that the top 20 estimators with the highest percentage of outlier generation have generally been proposed in recent years. It has been observed that the findings of Lukman and Ayinde (2017), who consider k_{70}, k_{73}, k_{84}, k_{86} and k_{88}, Lukman et al. (2017) who consider k_{144}, k_{162}, k_{163}, k_{164} and k_{166} and Muniz and Kibria (2009) who consider k_{22} in their studies, did not reach a conclusion that would reveal the superiority of these ridge parameter estimators. However, k_{11} proposed by Alkhamisi and Shukur (2007), k_{16} proposed by Dorugade (2014a), k_{39} proposed by Göktaş and Sevinç (2016), k_{90} proposed by Lukman and Ayinde (2017) and k_{145} and k_{147} proposed by Lukman et al. (2017) have been revealed to be more effective in some cases than the other estimators (either partially or fully) in the studies in which they were proposed.

We advise against using these ridge parameter estimators since they are among the top 20 estimators in our study in terms of outlier generation percentage for all cases considered. Likewise, although the k_{302} and k_{303} ridge parameter estimators proposed by Qasim et al. (2021) perform well in their proposed study for the beta ridge regression method, they are not deemed suitable for use in the multiple linear ridge regression method.

Lastly, the k_8 ridge parameter estimator proposed by Kibria (2003) was not recommended by the author of the study due to its high bias, despite its effectiveness in some cases compared to other estimators. Given the high percentage of outliers in this study, it is deemed inappropriate to use k_8 in line with the author's findings.

In terms of the normal distribution criterion, it was found that the estimation values generated by ridge parameter estimators k_8, k_{16}, k_{70}, k_{84}, k_{88}, k_{144}, k_{162}, k_{164}, k_{302}, k_{303}, and k_{331} did not provide normality in any case. With these estimators, the number of cases in which 60 of 366 ridge parameter estimators were normally distributed in all cases (6050) did not exceed 3025, whereas 185 estimators from the remaining estimators performed well by providing normality in nearly all cases. The estimators that generate data with normal distribution in all cases without exception are determined as k_{20}, k_{29}, k_{47}, k_{56}, k_{57}, k_{58}, k_{60}, k_{62}, k_{107}, k_{109}, k_{111}, k_{113}, k_{115}, k_{117}, k_{120}, k_{122}, k_{124}, k_{126}, k_{127}, k_{129}, k_{131}, k_{133}, k_{139}, k_{214}, k_{222}, k_{226}, k_{229}, k_{231}, k_{233}, k_{234}, k_{235}, k_{237}, k_{239}, k_{241}, k_{243}, k_{245}, k_{267}, k_{268}, k_{270}, k_{276}, k_{278}, k_{280}, k_{290} and k_{292}.

According to the results, it has been observed that while the majority of the estimators with high outlier generation lead to a low percentage of normality, estimators with a low percentage of outliers result in a high percentage of normality in general. All of these outcomes are statistically supported in the next section (see Table 4).

5.4. Spearman's rank correlation analysis between MSE, outlier detection, and normality criteria

Spearman's rank correlation coefficient values are presented in Table 4 to determine whether there is a linear interrelationship among the sum of rank numbers considered as a measure of MSE criterion, outlier generation ratios, and normality ratios.

According to the results presented in Table 4, there is a moderately positive significant correlation (0.127) between the sum of rank

Table 2
The 50 most efficient estimators in terms of MSE criterion based on the sum of rank numbers and the percentages of outlier and normality for these estimators.

Order Number	Ridge parameter estimator	Sum of rank numbers	Outlier percentage (%)	Normality percentage (%)	Order Number	Ridge parameter estimator	Sum of rank numbers	Outlier percentage (%)	Normality percentage (%)
1.	k_{robust}	255630	0.50	99.82	**26.**	k_{305}	578553	2.12	80.99
2.	k_{27}	307160	3.65	62.36	**27.**	k_{1}	585707	2.12	80.99
3.	k_{44}	348919	10.31	16.66	**28.**	k_{306}	592683	2.12	80.99
4.	k_{58}	355484	0.28	100	**29.**	k_{285}	596198	0.87	99.37
5.	k_{49}	386835	1.75	87.44	**30.**	k_{180}	597277	0.67	99.87
6.	k_{133}	422808	0.32	100	**31.**	k_{56}	597356	0.28	100
7.	k_{13}	425909	5.00	37.02	**32.**	k_{307}	598737	2.12	80.99
8.	k_{243}	475889	0.55	100	**33.**	k_{31}	602278	1.39	96.66
9.	k_{126}	485431	0.59	100	**34.**	k_{192}	602984	0.87	99.37
10.	k_{117}	492923	0.33	100	**35.**	k_{12}	603651	14.87	16.94
11.	k_{231}	494382	0.54	100	**36.**	k_{308}	604987	2.12	80.99
12.	k_{108}	517122	1.70	86.73	**37.**	k_{217}	608422	6.29	42.64
13.	k_{239}	522814	0.50	100	**38.**	k_{309}	611040	2.12	80.99
14.	k_{211}	522974	4.17	70.64	**39.**	k_{209}	614377	8.08	35.55
15.	k_{284}	535297	1.50	88.46	**40.**	k_{94}	614831	0.70	99.80
16.	k_{29}	535703	1.01	100	**41.**	k_{310}	617093	2.12	80.99
17.	k_{190}	539115	1.47	88.89	**42.**	k_{157}	617690	0.83	99.60
18.	k_{143}	545574	2.03	85.36	**43.**	k_{110}	619106	0.97	99.40
19.	k_{151}	551259	1.39	98.64	**44.**	k_{26}	622471	5.50	5.24
20.	k_{42}	554228	2.12	80.99	**45.**	k_{5}	623708	2.12	80.99
21.	k_{10}	564641	2.12	80.99	**46.**	k_{127}	628412	0.61	100
22.	k_{83}	567144	2.03	85.36	**47.**	k_{312}	631512	2.14	80.51
23.	k_{24}	569071	5.04	18.91	**48.**	k_{257}	633005	2.38	73.12
24.	k_{249}	571586	2.40	71.79	**49.**	k_{221}	641548	2.05	84.21
25.	k_{41}	575275	2.12	80.99	**50.**	k_{155}	641875	3.39	72.61

Table 3
Top 20 ridge parameter estimators with the lowest and highest outlier percentages.

Order number	The 20 worst ridge parameter estimators by percentage of outliers		The 20 best ridge parameter estimators by percentage of outliers	
	Estimators	Percentages (%)	Estimators	Percentages (%)
1.	k_{84}	25.82	k_{56}	0.28
2.	k_{144}	25.77	k_{58}	0.28
3.	k_{70}	25.76	k_{241}	0.29
4.	k_{162}	25.76	k_{237}	0.29
5.	k_{302}	25.34	k_{245}	0.29
6.	k_{88}	24.44	k_{233}	0.30
7.	k_{8}	24.31	k_{20}	0.30
8.	k_{164}	24.31	k_{267}	0.30
9.	k_{16}	24.30	k_{120}	0.31
10.	k_{303}	23.52	k_{229}	0.31
11.	k_{39}	23.35	k_{270}	0.31
12.	k_{11}	17.25	k_{129}	0.32
13.	k_{22}	16.58	k_{133}	0.32
14.	k_{163}	16.58	k_{113}	0.33
15.	k_{145}	16.57	k_{117}	0.33
16.	k_{86}	16.53	k_{62}	0.34
17.	k_{73}	16.22	k_{282}	0.34
18.	k_{166}	16.21	k_{111}	0.35
19.	k_{147}	16.18	k_{115}	0.35
20.	k_{90}	16.13	k_{214}	0.37

Table 4
Spearman's rank correlation analysis among performance criteria.

		Sum of rank numbers	Outlier ratios	Normality ratios
Sum of rank numbers	Correlation coefficient	1	0.127	−0.145
	Sig. (2-tailed)		0.015*	0.005*
Outlier ratios	Correlation coefficient	0.127	1	−0.970
	Sig. (2-tailed)	0.015*		<0.0001*
Normality ratios	Correlation coefficient	−0.145	−0.970	1
	Sig. (2-tailed)	0.005*	<0.0001*	

*Significant at 0.05 significance level.

numbers and outlier generation ratios of ridge parameter estimators ($p < 0.05$). On the other hand, there is a moderately negative significant correlation (−0.145) between the sum of rank numbers and normality ratios ($p < 0.05$). Additionally, a significant inverse correlation (−0.970) was found between outlier generation ratios and normality ratios ($p < 0.05$). This means that a decrease in outlier ratios or an increase in normality ratios will improve the efficiency of ridge parameter estimators based on the MSE criterion, leading to a significant decrease in the sum of rank numbers. Similarly, it can be said that as the outlier generation ratios decrease, the normality performance of ridge parameter estimators will also improve significantly.

The normality tests after removing outliers from the values generated by 366 estimators and k_{robust} using the MAD-based outlier detection method revealed that the values generated by all other estimators, except k_{27}, k_{39}, k_{55} and k_{302}, were normally distributed for all cases. Their normality ratios ranged from 99.55% to 99.98%.

6. Discussion and conclusions

The existence of a perfect or near-linear relationship among predictors in a multiple linear regression model (multicollinearity) is a common issue in practice. It hinders obtaining reliable statistical inferences regarding regression parameters obtained by the least squares method. In other words, if the predictors are multicollinear, they lead to over-inflation of the variance values of the regression coefficients. This may cause model tests to be insignificant, wide confidence intervals, and some regression coefficients to be statistically insignificant with an opposite sign. The ridge regression method is a commonly used method for dealing with multicollinearity. A total of 366 different estimators proposed in the literature for estimating the ridge parameter have been identified within the scope of this method. The common feature of these estimators is that their performance has mostly been evaluated based on only the MSE criterion, within a narrow scope (less than 200 cases in total) in terms of number of predictors, sample size, correlation level, and errors' standard deviation. The goal of this study, using the "search method" proposed by Göktaş et al. (2021), is to develop a new robust ridge parameter estimator having minimum MSE, low outlier generation, and assuring normality in most of the 6050 cases considered widely

$k_1 : \frac{p\widehat{\sigma}^2}{\widehat{\beta}'\widehat{\beta}}$ Hoerl et al. (1975) $k_2 : \frac{p\widehat{\sigma}^2}{\sum_{i=1}^{p}\lambda_i\widehat{a}_i^2}$ Lawless and Wang (1976) $k_3 : p\widehat{\sigma}^2 / \sum_{i=1}^{p}\left\{\widehat{a}_i^2 / \left[1+\left(1+\lambda_i\left(\frac{\widehat{a}_i^2}{\widehat{\sigma}^2}\right)^{\frac{1}{2}}\right)\right]\right\}$ Nomura (1988)

$k_4 : (\lambda_{max}\widehat{\sigma}^2) / ((n-p-1)\widehat{\sigma}^2 + \lambda_{max}\widehat{\beta}_{max}^2)$ Khalaf and Shukur (2005)

k_5:$\max\left(0, \frac{p\widehat{\sigma}^2}{\widehat{\alpha}'\widehat{\alpha}} - \frac{1}{n(VIF_j)_{max}}\right)$ where $VIF_j = \frac{1}{1-R_j^2}\, j = 1, 2, \cdots, p$ is the variance inflation factor of jth regressor. Dorugade and Kashid (2010)

$k_6 : \frac{\widehat{\sigma}^2}{(\prod_{i=1}^{p}\widehat{a}_i^2)^{\frac{1}{p}}}$ $k_7 : med\left(\frac{\widehat{\sigma}^2}{\widehat{a}_i^2}\right)\ i = 1, 2, \cdots, p$ $k_8 : \frac{1}{p}\sum_{i=1}^{p}\frac{\widehat{\sigma}^2}{\widehat{a}_i^2}$ $k_9 : \frac{\widehat{\sigma}^2}{\widehat{a}_{max}^2}$ Hoerl and Kennard (1970)

Kibria (2003)

$k_{10} : \frac{p\widehat{\sigma}^2}{\widehat{\alpha}'\widehat{\alpha}} + \frac{1}{\lambda_{max}} = K1 + \frac{1}{\lambda_{max}}$ $k_{11} : max\left(\frac{\widehat{\sigma}^2}{\widehat{\beta}_i^2} + \frac{1}{\lambda_i}\right)$ $k_{12} : \frac{1}{p}\sum_{i=1}^{p}\left(\frac{\widehat{\sigma}^2}{\widehat{\beta}_i^2} + \frac{1}{\lambda_i}\right)$ $k_{13} : med\left(\frac{\widehat{\sigma}^2}{\widehat{\beta}_i^2} + \frac{1}{\lambda_i}\right)$

$k_{14} : \frac{p\widehat{\sigma}^2}{\sum_{i=1}^{p}\lambda_i\widehat{\beta}_i^2} + \frac{1}{\lambda_{max}}$ $k_{15} : \frac{\widehat{\sigma}^2}{\widehat{\beta}_{max}^2} + \frac{1}{\lambda_{max}}\ \mathrm{i} = 1, 2, \cdots, p$ Alkhamisi and Shukur (2007)

$k_i\ (\mathrm{AD}) = \frac{2\widehat{\sigma}^2}{\lambda_{max}\widehat{a}_i^2},\ \mathrm{i} = 1, 2, \cdots, p$ k_{16}:Arithmetic Mean$(k_i(\mathrm{AD})) = \frac{2}{p\lambda_{max}}\sum_{i=1}^{p}\frac{\widehat{\sigma}^2}{\widehat{a}_i^2}$

$k_{17} : med(k_i(\mathrm{AD})) = med\left(\frac{2\widehat{\sigma}^2}{\lambda_{max}\widehat{a}_i^2}\right)$ k_{18}:Geometric Mean $(k_i(\mathrm{AD})) = \frac{2\widehat{\sigma}^2}{\lambda_{max}(\prod_{i=1}^{p}\widehat{a}_i^2)^{\frac{1}{p}}}$

k_{19}:Harmonic Mean $(k_i(\mathrm{AD})) = \frac{2p}{\lambda_{max}}\sum_{i=1}^{p}\frac{\widehat{\sigma}^2}{\widehat{a}_i^2}$ Dorugade (2014a)

$k_{20} : \left(\prod_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2 + \lambda_i\widehat{a}_i^2}\right)^{\frac{1}{p}} m_i = \sqrt{\frac{\widehat{\sigma}^2}{\widehat{a}_i^2}}$ $k_{21} : max\left(\frac{1}{m_i}\right)$ $k_{22} : max(m_i)$ $k_{23} : \left(\prod_{i=0}^{p}\frac{1}{m_i}\right)^{\left(\frac{1}{p}\right)}$ $k_{24} : (\prod_{i=0}^{p}m_i)^{\left(\frac{1}{p}\right)}$ $k_{25} : med\left(\frac{1}{m_i}\right)$ $k_{26} : med(m_i)$ Muniz and Kibria (2009)

$q_i = \frac{\lambda_{max}}{(n-p)\widehat{\sigma}^2 + \lambda_{max}\widehat{a}_i^2}$ $k_{27} : max\left(\frac{1}{q_i}\right)$ $k_{28} : max(q_i)$ $k_{29} : \prod_{i=0}^{p}\left(\frac{1}{q_i}\right)^{\frac{1}{p}}$ $k_{30} : \prod_{i=0}^{p}(q_i)^{\frac{1}{p}}$ $k_{31} : med\left(\frac{1}{q_i}\right)$ $k_{32} : med(q_i)$ Kibria, Mansson, and Shukur (2012)

$k_{33} : \frac{p^2}{\lambda_{max}^2}\frac{\widehat{\sigma}^2}{\sum_{i=1}^{p}\widehat{a}_i^2}$ $k_{34} : \frac{p^3}{\lambda_{max}^3}\frac{\widehat{\sigma}^2}{\sum_{i=1}^{p}\widehat{a}_i^2}$ $k_{35} : \frac{P}{\frac{1}{\lambda_{max}^3}}\frac{\widehat{\sigma}^2}{\sum_{i=1}^{p}\widehat{a}_i^2}$ $k_{36} : \frac{P}{(\sum_{i=1}^{p}\sqrt{\lambda_i})^{\frac{1}{3}}}\frac{\widehat{\sigma}^2}{\sum_{i=1}^{p}\widehat{a}_i^2}$ $k_{37} : \frac{2p}{\sqrt{\lambda_{max}}}\frac{\widehat{\sigma}^2}{\sum_{i=1}^{p}\widehat{a}_i^2}$ Asar, Karaibrahimoğlu, and Genç (2014)

$k_{38} : \sqrt{med(m_i)}$ $k_{39} : \frac{\widehat{\sigma}^2}{(med(\widehat{a}_i))^2}$ Göktaş and Sevinç (2016)

$k_{40} : \widehat{\sigma}^2\frac{\sum_{i=1}^{p}(\lambda_i\widehat{a}_i)^2}{(\sum_{i=1}^{p}\lambda_i\widehat{a}_i^2)^2}$ Hocking, Speed, and Lynn (1976)

$k_{41} : \frac{p\widehat{\sigma}^2}{\widehat{\beta}'\widehat{\beta}} + \frac{1}{\lambda_{max}\widehat{\alpha}'\widehat{\alpha}}$ $k_{42} : \frac{p\widehat{\sigma}^2}{\widehat{\beta}'\widehat{\beta}} + \frac{1}{2(\sqrt{\lambda_{max}/\lambda_{min}})^2}$ Bhat and Vidya (2016)

$k_{43} : \widehat{\sigma}^2\frac{\sum_{i=1}^{p}(\lambda_i\widehat{a}_i)^2}{(\sum_{i=1}^{p}\lambda_i\widehat{a}_i^2)^2} + \frac{1}{\lambda_{max}}$ Al-Hassan (2010)

$k_{44} : max\left(\frac{S^2}{\widehat{\beta}_i}\right)$ $k_{45} : max\left(\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2 + \lambda_i\widehat{\beta}_i^2}\right)$ $k_{46} : \frac{1}{p}\sum_{i=1}^{p}\left(\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2 + \lambda_i\widehat{\beta}_i^2}\right)$ $k_{47} : med\left(\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2 + \lambda_i\widehat{\beta}_i^2}\right)$

Alkhamisi et al. (2006)

$k_{48} : \frac{p\widehat{\sigma}^2}{\widehat{\alpha}'\widehat{\alpha}}$ where $\widehat{\sigma}^2 = \frac{RSS}{n-p+2}$ Khalaf (2012)

$k_{49} : \frac{\widehat{\sigma}^2}{min\left(\frac{\widehat{\sigma}^2}{\widehat{a}_i^2} + \frac{1}{\lambda_i}\right)}$ Dorugade (2014b)

$k_{50} : \frac{(\lambda_{max}+\lambda_{min})}{2\sum_{i=1}^{n}|\widehat{\beta}_i|}\frac{p\widehat{\sigma}^2}{\widehat{\beta}'\widehat{\beta}}$ $k_{51} : \frac{(\lambda_{max}+\lambda_{min})}{2\sum_{i=1}^{n}|\widehat{\beta}_i|}\frac{(\lambda_{max}\widehat{\sigma}^2)}{((n-p-1)\widehat{\sigma}^2 + \lambda_{max}\widehat{\beta}_{max}^2)}$ Khalaf (2013)

$k_{52} : \frac{\widehat{\sigma}^2}{2}\left(\frac{1}{\widehat{\beta}_{max}^2} + \frac{p}{\widehat{\beta}'\widehat{\beta}}\right)$ $k_{53} : \frac{2\widehat{\sigma}^2}{\widehat{\beta}_{max}^2 + \frac{\widehat{\beta}'\widehat{\beta}}{p}}$ $k_{54} : \widehat{\sigma}^2\sqrt{\frac{P}{\widehat{\beta}_{max}^2\widehat{\beta}'\widehat{\beta}}}$ $k_{55} :$

$$\begin{cases} \text{The sum of } \left(\frac{\widehat{\sigma}^2}{\widehat{a}_{max}^2}, \frac{p\widehat{\sigma}^2}{\widehat{\beta}'\widehat{\beta}}\right) & \text{if } \frac{\widehat{\sigma}^2}{\widehat{a}_{max}^2} + \frac{p\widehat{\sigma}^2}{\widehat{\beta}'\widehat{\beta}} < 1 \\ \text{The sum of } \left(\frac{\widehat{\sigma}^2}{\widehat{a}_{max}^2}, \frac{p\widehat{\sigma}^2}{\widehat{\beta}'\widehat{\beta}}\right) \Big/ 2 & \text{if } \frac{\widehat{\sigma}^2}{\widehat{a}_{max}^2} + \frac{p\widehat{\sigma}^2}{\widehat{\beta}'\widehat{\beta}} \geq 1 \end{cases}$$

Khalaf and Iguernane (2014)

$k_{56} : \sigma, \widehat{\sigma}^2 = \frac{Y'Y - \widehat{\alpha}'Z'Y}{n-p-1}$ Dorugade (2016)

$k_{57} : \sqrt{\frac{(\lambda_{max}\widehat{\sigma}^2)}{((n-p-1)\widehat{\sigma}^2 + \lambda_{max}\widehat{\beta}_{max}^2)}}$

Khalaf and Iguernane (2016)

$k_{58} : p\sigma, \widehat{\sigma}^2 = \frac{Y'Y - \widehat{\alpha}'Z'Y}{n-p-1}$ Lukman and Olatunji (2018)

$k_{59} : max\left(1/\sqrt{\frac{(\lambda_{max}\widehat{\sigma}^2)}{((n-p)\widehat{\sigma}^2 + \lambda_{max}\widehat{a}_i^2)}}\right)$ $k_{60} : max\left(\sqrt{\frac{(\lambda_{max}\widehat{\sigma}^2)}{((n-p)\widehat{\sigma}^2 + \lambda_{max}\widehat{a}_i^2)}}\right)$ $k_{61} : \left(\prod_{i=1}^{P}1/\sqrt{\frac{(\lambda_{max}\widehat{\sigma}^2)}{((n-p)\widehat{\sigma}^2 + \lambda_{max}\widehat{a}_i^2)}}\right)^{\frac{1}{p}}$

$k_{62} : \left(\prod_{i=1}^{p}\sqrt{\frac{(\lambda_{max}\widehat{\sigma}^2)}{((n-p)\widehat{\sigma}^2 + \lambda_{max}\widehat{a}_i^2)}}\right)^{\frac{1}{p}}$ $k_{63} : med\left(1/\sqrt{\frac{(\lambda_{max}\widehat{\sigma}^2)}{((n-p)\widehat{\sigma}^2 + \lambda_{max}\widehat{a}_i^2)}}\right)$ Muniz et al. (2012)

(continued on next page)

(*continued*)

$$k_{64}:\frac{\sqrt{5p}}{\lambda_{max}}\frac{\widehat{\sigma}^2}{\sum_{i=1}^{p}\widehat{a}_i^2}\ k_{65}:\frac{p}{\sqrt{\lambda_{max}}}\frac{\widehat{\sigma}^2}{\sum_{i=1}^{p}\widehat{a}_i^2}\ k_{66}:\frac{2p}{\sum_{i=1}^{p}(\lambda_i^{1/4})}\frac{\widehat{\sigma}^2}{\sum_{i=1}^{p}\widehat{a}_i^2}\ k_{67}:\frac{2p}{\sqrt{\sum_{i=1}^{p}\lambda_i}}\frac{\widehat{\sigma}^2}{\sum_{i=1}^{p}\widehat{a}_i^2}$$ Karaibrahimoğlu et al. (2016)

$$\widehat{K}_{HK}^{FMO}=\frac{\widehat{\sigma}^2}{Max(\widehat{a}_i^2)}$$ Hoerl and Kennard (1970) $k_{68}:\frac{1}{\widehat{K}_{HK}^{FMO}}\ k_{69}:\sqrt{\widehat{K}_{HK}^{FMO}}\ k_{70}:\max\left(\frac{\widehat{\sigma}^2}{\widehat{a}_i^2}\right)\ k_{71}:max\left(\frac{1}{\widehat{\sigma}^2/\widehat{a}_i^2}\right)$

$$\widehat{K}_{HK}^{AMO}=\frac{1}{p}\sum_{i=1}^{p}\frac{\widehat{\sigma}^2}{\widehat{a}_i^2}$$ Kibria (2003) $k_{72}:\frac{1}{\widehat{K}_{HK}^{AMO}}\ k_{73}:\sqrt{\widehat{K}_{HK}^{AMO}}\ k_{74}:\frac{1}{\sqrt{\widehat{K}_{HK}^{AMO}}}$

$$\widehat{K}_{HK}^{HMO}=\frac{p\widehat{\sigma}^2}{\widehat{\beta}\,\widehat{\beta}}$$ Hoerl et al. (1975) $k_{75}:\frac{1}{\widehat{K}_{HK}^{HMO}}\ k_{76}:\sqrt{\widehat{K}_{HK}^{HMO}}\ k_{77}:\frac{1}{\sqrt{\widehat{K}_{HK}^{HMO}}}\ k_{78}:\frac{1}{\widehat{\sigma}^2/(\prod_{i=1}^{p}\widehat{a}_i^2)^{\frac{1}{p}}}\ k_{79}:med\left(\frac{1}{\widehat{\sigma}^2/\widehat{a}_i^2}\right)\ k_{80}:\frac{\widehat{\sigma}^2}{\max(\lambda_i)max(\widehat{a}_i^2)}\ k_{81}:\frac{1}{\frac{\widehat{\sigma}^2}{\max(\lambda_i)max(\widehat{a}_i^2)}}\ k_{82}:\sqrt{\frac{\widehat{\sigma}^2}{\max(\lambda_i)max(\widehat{a}_i^2)}}\ k_{83}:$$

$$\frac{1}{\sqrt{\frac{\widehat{\sigma}^2}{\max(\lambda_i)max(\widehat{a}_i^2)}}}\ k_{84}:max\left(\frac{\widehat{\sigma}^2}{\lambda_i\widehat{a}_i^2}\right)\ k_{85}:max\left(\frac{1}{\frac{\widehat{\sigma}^2}{\lambda_i\widehat{a}_i^2}}\right)\ k_{86}:max\left(\sqrt{\frac{\widehat{\sigma}^2}{\lambda_i\widehat{a}_i^2}}\right)\ k_{87}:max\left(\frac{1}{\sqrt{\frac{\widehat{\sigma}^2}{\lambda_i\widehat{a}_i^2}}}\right)\ k_{88}:\frac{1}{p}\sum_{i=1}^{p}\frac{\widehat{\sigma}^2}{\lambda_i\widehat{a}_i^2}\ k_{89}:\frac{1}{\frac{1}{p}\sum_{i=1}^{p}\frac{\widehat{\sigma}^2}{\lambda_i\widehat{a}_i^2}}\ k_{90}:\sqrt{\frac{1}{p}\sum_{i=1}^{p}\frac{\widehat{\sigma}^2}{\lambda_i\widehat{a}_i^2}}\ k_{91}:\frac{1}{\sqrt{\frac{1}{p}\sum_{i=1}^{p}\frac{\widehat{\sigma}^2}{\lambda_i\widehat{a}_i^2}}}\ k_{92}:$$

$$\frac{1}{\frac{p\widehat{\sigma}^2}{\sum_{i=1}^{p}\lambda_i\widehat{a}_i^2}}\ k_{93}:\sqrt{\frac{p\widehat{\sigma}^2}{\sum_{i=1}^{p}\lambda_i\widehat{a}_i^2}}\ k_{94}:\frac{1}{\sqrt{\frac{p\widehat{\sigma}^2}{\sum_{i=1}^{p}\lambda_i\widehat{a}_i^2}}}\ k_{95}:\frac{\widehat{\sigma}^2}{(\prod_{i=1}^{p}\lambda_i\widehat{a}_i^2)^{1/p}}\ k_{96}:\frac{1}{\frac{\widehat{\sigma}^2}{(\prod_{i=1}^{p}\lambda_i\widehat{a}_i^2)^{1/p}}}\ k_{97}:\sqrt{\frac{\widehat{\sigma}^2}{(\prod_{i=1}^{p}\lambda_i\widehat{a}_i^2)^{1/p}}}\ k_{98}:\frac{1}{\sqrt{\frac{\widehat{\sigma}^2}{(\prod_{i=1}^{p}\lambda_i\widehat{a}_i^2)^{1/p}}}}\ k_{99}:med\left(\frac{\widehat{\sigma}^2}{\lambda_i\widehat{a}_i^2}\right)\ k_{100}:med\left(\frac{1}{\frac{\widehat{\sigma}^2}{\lambda_i\widehat{a}_i^2}}\right)\ k_{101}:med\left(\sqrt{\frac{\widehat{\sigma}^2}{\lambda_i\widehat{a}_i^2}}\right)\ k_{102}:$$

$$med\left(\frac{1}{\sqrt{\frac{\widehat{\sigma}^2}{\lambda_i\widehat{a}_i^2}}}\right)\ k_{103}:\frac{\widehat{\sigma}^2\ max(\lambda_i)}{(n-p)\widehat{\sigma}^2+max(\lambda_i)max(\widehat{a}_i^2)}\ k_{104}:\frac{1}{\frac{\widehat{\sigma}^2\ max(\lambda_i)}{(n-p)\widehat{\sigma}^2+max(\lambda_i)max(\widehat{a}_i^2)}}\ k_{105}:\sqrt{\frac{\widehat{\sigma}^2\ max(\lambda_i)}{(n-p)\widehat{\sigma}^2+max(\lambda_i)max(\widehat{a}_i^2)}}\ k_{106}:\frac{1}{\sqrt{\frac{\widehat{\sigma}^2\ max(\lambda_i)}{(n-p)\widehat{\sigma}^2+max(\lambda_i)max(\widehat{a}_i^2)}}}\ k_{107}:$$

$$max\left(\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{a}_i^2}\right)\ k_{108}:max\left(\frac{1}{\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{a}_i^2}}\right)\ k_{109}:max\left(\sqrt{\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{a}_i^2}}\right)\ k_{110}:max\left(\frac{1}{\sqrt{\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{a}_i^2}}}\right)\ k_{111}:\frac{1}{p}\sum_{i=1}^{p}\left(\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{a}_i^2}\right)\ k_{112}:$$

$$\frac{1}{\frac{1}{p}\sum_{i=1}^{p}\left(\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{a}_i^2}\right)}\ k_{113}:\sqrt{\frac{1}{p}\sum_{i=1}^{p}\left(\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{a}_i^2}\right)}\ k_{114}:\frac{1}{\sqrt{\frac{1}{p}\sum_{i=1}^{p}\left(\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{a}_i^2}\right)}}\ k_{115}:p\sum_{i=1}^{p}\left(\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{a}_i^2}\right)\ k_{116}:\frac{1}{p\sum_{i=1}^{p}\left(\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{a}_i^2}\right)}\ k_{117}:$$

$$\sqrt{p\sum_{i=1}^{p}\left(\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{a}_i^2}\right)}\ k_{118}:\frac{1}{\sqrt{p\sum_{i=1}^{p}\left(\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{a}_i^2}\right)}}\ k_{119}:\frac{1}{\left(\prod_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{a}_i^2}\right)^{1/p}}\ k_{120}:\sqrt{\left(\prod_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{a}_i^2}\right)^{1/p}}\ k_{121}:\frac{1}{\sqrt{\left(\prod_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{a}_i^2}\right)^{1/p}}}\ k_{122}:$$

$$med\left(\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{a}_i^2}\right)\ k_{123}:med\left(\frac{1}{\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{a}_i^2}}\right)\ k_{124}:med\left(\sqrt{\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{a}_i^2}}\right)\ k_{125}:med\left(\frac{1}{\sqrt{\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{a}_i^2}}}\right)\ k_{126}:max\left(\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+max(\lambda_i)\widehat{a}_i^2}\right)\ k_{127}:$$

$$\frac{1}{p}\sum_{i=1}^{p}\left(\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+max(\lambda_i)\widehat{a}_i^2}\right)\ k_{128}:\frac{1}{\frac{1}{p}\sum_{i=1}^{p}\left(\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+max(\lambda_i)\widehat{a}_i^2}\right)}\ k_{129}:\sqrt{\frac{1}{p}\sum_{i=1}^{p}\left(\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+max(\lambda_i)\widehat{a}_i^2}\right)}\ k_{130}:\frac{1}{\sqrt{\frac{1}{p}\sum_{i=1}^{p}\left(\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+max(\lambda_i)\widehat{a}_i^2}\right)}}\ k_{131}:$$

$$p\sum_{i=1}^{p}\left(\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+max(\lambda_i)\widehat{a}_i^2}\right)\ k_{132}:\frac{1}{p\sum_{i=1}^{p}\left(\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+max(\lambda_i)\widehat{a}_i^2}\right)}\ k_{133}:\sqrt{p\sum_{i=1}^{p}\left(\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+max(\lambda_i)\widehat{a}_i^2}\right)}\ k_{134}:\frac{1}{\sqrt{p\sum_{i=1}^{p}\left(\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+max(\lambda_i)\widehat{a}_i^2}\right)}}\ k_{135}:$$

$$\left(\prod_{i=1}^{p}\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+max(\lambda_i)\widehat{a}_i^2}\right)^{1/p}\ k_{136}:\frac{1}{\left(\prod_{i=1}^{p}\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+max(\lambda_i)\widehat{a}_i^2}\right)^{1/p}}\ k_{137}:med\left(\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+max(\lambda_i)\widehat{a}_i^2}\right)\ k_{138}:med\left(\frac{1}{\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+max(\lambda_i)\widehat{a}_i^2}}\right)\ k_{139}:$$

$$med\left(\sqrt{\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+max(\lambda_i)\widehat{a}_i^2}}\right)$$

Lukman and Ayinde (2017)

$$k_{140}:\frac{2\widehat{\sigma}^2}{max(\lambda_i)max(\widehat{a}_i^2)}\ k_{141}:\frac{1}{\frac{2\widehat{\sigma}^2}{max(\lambda_i)max(\widehat{a}_i^2)}}\ k_{142}:\sqrt{\frac{2\widehat{\sigma}^2}{max(\lambda_i)max(\widehat{a}_i^2)}}\ k_{143}:\frac{1}{\sqrt{\frac{2\widehat{\sigma}^2}{max(\lambda_i)max(\widehat{a}_i^2)}}}\ k_{144}:max\left(\frac{2\widehat{\sigma}^2}{max(\lambda_i)\widehat{a}_i^2}\right)\ k_{145}:max\left(\sqrt{\frac{2\widehat{\sigma}^2}{max(\lambda_i)\widehat{a}_i^2}}\right)\ k_{146}:\frac{1}{\frac{2\widehat{\sigma}^2}{max(\lambda_i)p}\sum_{i=1}^{p}\frac{1}{\widehat{a}_i^2}}\ k_{147}:$$

$$\sqrt{\frac{2\widehat{\sigma}^2}{max(\lambda_i)p}\sum_{i=1}^{p}\frac{1}{\widehat{a}_i^2}}\ k_{148}:\frac{1}{\sqrt{\frac{2\widehat{\sigma}^2}{max(\lambda_i)p}\sum_{i=1}^{p}\frac{1}{\widehat{a}_i^2}}}\ k_{149}:\frac{1}{\frac{2p\widehat{\sigma}^2}{max(\lambda_i)\sum_{i=1}^{p}\widehat{a}_i^2}}\ k_{150}:\sqrt{\frac{2p\widehat{\sigma}^2}{max(\lambda_i)\sum_{i=1}^{p}\widehat{a}_i^2}}\ k_{151}:\frac{1}{\sqrt{\frac{2p\widehat{\sigma}^2}{max(\lambda_i)\sum_{i=1}^{p}\widehat{a}_i^2}}}\ k_{152}:\frac{1}{\frac{2\widehat{\sigma}^2}{max(\lambda_i)(\prod_{i=1}^{p}\widehat{a}_i^2)^{1/p}}}\ k_{153}:\sqrt{\frac{2\widehat{\sigma}^2}{max(\lambda_i)(\prod_{i=1}^{p}\widehat{a}_i^2)^{1/p}}}\ k_{154}:$$

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$$k_{155}:\frac{1}{\sqrt{\frac{2\widehat{\sigma}^2}{\max(\lambda_i)(\prod_{i=1}^{p}\widehat{a}_i^2)^{1/p}}}}\quad k_{156}:med\left(\frac{1}{\frac{2\widehat{\sigma}^2}{\max(\lambda_i)\widehat{a}_i^2}}\right)\quad k_{157}:med\left(\sqrt{\frac{2\widehat{\sigma}^2}{\max(\lambda_i)\widehat{a}_i^2}}\right)\quad k_{158}:med\left(\frac{1}{\sqrt{\frac{2\widehat{\sigma}^2}{\max(\lambda_i)\widehat{a}_i^2}}}\right)$$

$$k_{159}:\frac{2\widehat{\sigma}^2}{\sqrt{\sum_{i=1}^{p}\lambda_i\max(\widehat{a}_i^2)}}\quad k_{160}:\frac{1}{\frac{2\widehat{\sigma}^2}{\sqrt{\sum_{i=1}^{p}\lambda_i}\max(\widehat{a}_i^2)}}\quad k_{161}:\sqrt{\frac{2\widehat{\sigma}^2}{\sqrt{\sum_{i=1}^{p}\lambda_i}\max(\widehat{a}_i^2)}}$$

$$k_{162}:\frac{1}{\sqrt{\frac{2\widehat{\sigma}^2}{\sqrt{\sum_{i=1}^{p}\lambda_i}\max(\widehat{a}_i^2)}}}\quad k_{163}:max\left(\frac{2\widehat{\sigma}^2}{\sqrt{\sum_{i=1}^{p}\lambda_i}\,\widehat{a}_i^2}\right)\quad k_{164}:max\left(\sqrt{\frac{2\widehat{\sigma}^2}{\sqrt{\sum_{i=1}^{p}\lambda_i}\,\widehat{a}_i^2}}\right)\quad k_{165}:\frac{2\widehat{\sigma}^2}{\sqrt{\sum_{i=1}^{p}\lambda_i}\,p}\sum_{i=1}^{p}\frac{1}{\widehat{a}_i^2}$$

$$k_{166}:\frac{1}{\frac{2\widehat{\sigma}^2}{\sqrt{\sum_{i=1}^{p}\lambda_i}\,p}\sum_{i=1}^{p}\frac{1}{\widehat{a}_i^2}}\quad k_{167}:\sqrt{\frac{2\widehat{\sigma}^2}{\sqrt{\sum_{i=1}^{p}\lambda_i}\,p}\sum_{i=1}^{p}\frac{1}{\widehat{a}_i^2}}$$

$$k_{168}:\frac{1}{\sqrt{\frac{2\widehat{\sigma}^2}{\sqrt{\sum_{i=1}^{p}\lambda_i}\,p}\sum_{i=1}^{p}\frac{1}{\widehat{a}_i^2}}}\quad k_{169}:\frac{1}{\frac{2p\widehat{\sigma}^2}{\sqrt{\sum_{i=1}^{p}\lambda_i}\;\sum_{i=1}^{p}\widehat{a}_i^2}}\quad k_{170}:\sqrt{\frac{2p\widehat{\sigma}^2}{\sqrt{\sum_{i=1}^{p}\lambda_i}\;\sum_{i=1}^{p}\widehat{a}_i^2}}\quad k_{171}:\frac{1}{\sqrt{\frac{2p\widehat{\sigma}^2}{\sqrt{\sum_{i=1}^{p}\lambda_i}\;\sum_{i=1}^{p}\widehat{a}_i^2}}}$$

$$k_{172}:\frac{2\widehat{\sigma}^2}{\sqrt{\sum_{i=1}^{p}\lambda_i}\,(\prod_{i=1}^{p}\widehat{a}_i^2)^{1/p}}\quad k_{173}:\frac{1}{\frac{2\widehat{\sigma}^2}{\sqrt{\sum_{i=1}^{p}\lambda_i}\,(\prod_{i=1}^{p}\widehat{a}_i^2)^{1/p}}}\quad k_{174}:\sqrt{\frac{2\widehat{\sigma}^2}{\sqrt{\sum_{i=1}^{p}\lambda_i}\,(\prod_{i=1}^{p}\widehat{a}_i^2)^{1/p}}}$$

$$k_{175}:\frac{1}{\sqrt{\frac{2\widehat{\sigma}^2}{\sqrt{\sum_{i=1}^{p}\lambda_i}\,(\prod_{i=1}^{p}\widehat{a}_i^2)^{1/p}}}}\quad k_{176}:med\left(\frac{2\widehat{\sigma}^2}{\sqrt{\sum_{i=1}^{p}\lambda_i}\,\widehat{a}_i^2}\right)\quad k_{177}:med\left(\frac{1}{\frac{2\widehat{\sigma}^2}{\sqrt{\sum_{i=1}^{p}\lambda_i}\,\widehat{a}_i^2}}\right)\quad k_{178}:med\left(\sqrt{\frac{2\widehat{\sigma}^2}{\sqrt{\sum_{i=1}^{p}\lambda_i}\,\widehat{a}_i^2}}\right)$$

$$k_{179}:med\left(\frac{1}{\sqrt{\frac{2\widehat{\sigma}^2}{\sqrt{\sum_{i=1}^{p}\lambda_i}\,\widehat{a}_i^2}}}\right)$$

Lukman et al. (2017)

$$k_{179}:\frac{1}{p}\sum_{i=1}^{p}\sqrt{\frac{\widehat{\sigma}^2}{\lambda_i\widehat{a}_i^2}}\quad k_{180}:max\left(\frac{1}{\sqrt{\frac{\widehat{\sigma}^2}{\lambda_i\widehat{a}_i^2}}}\right)\quad k_{181}:\frac{1}{p}\sum_{i=1}^{p}\frac{1}{\sqrt{\frac{\widehat{\sigma}^2}{\lambda_i\widehat{a}_i^2}}}\quad k_{182}:\frac{p}{\sum_{i=1}^{p}\sqrt{\frac{\lambda_i\widehat{a}_i^2}{\widehat{\sigma}^2}}}\quad k_{183}:\frac{p}{\sum_{i=1}^{p}\sqrt{\frac{\widehat{\sigma}^2}{\lambda_i\widehat{a}_i^2}}}$$

Asar and Genç (2017)

$$k_{184}:\frac{np\widehat{\sigma}^2}{(p\max(\widehat{a}_i^2)\sum_{i=1}^{p}\widehat{a}_i^2\sum_{i=1}^{p}\lambda_i\widehat{a}_i^2)^{1/p}}\qquad k_{185}:\frac{p\widehat{\sigma}^2}{\sum_{i=1}^{p}(2\widehat{a}_i^2+(\widehat{\sigma}^2/\lambda_i))}\quad k_{186}:min\left(\frac{\widehat{\sigma}^2}{2\widehat{a}_i^2+(\widehat{\sigma}^2/\lambda_i)}\right)$$

Haggag (2016)

Kibria and Lukman (2020)

$$k_{187}:\frac{(p-2)\widehat{\sigma}^2}{\sum_{i=1}^{p}\widehat{\beta}_i^2}\qquad k_{188}:\frac{n(p-2)\widehat{\sigma}^2}{\widehat{\beta}^TX^TX\widehat{\beta}}\qquad k_{189}:\frac{p\widehat{\sigma}^2}{\sum_{i=1}^{p}[\widehat{a}_i^2/\{(\widehat{a}_i^4\lambda_i^2/4\widehat{\sigma}^4+6\widehat{a}_i^2\lambda_i/\widehat{\sigma}^2)^{1/2}-\widehat{a}_i^2\lambda_i/2\widehat{\sigma}^2\}]}$$

Thisted (1976)

Venables and Ripley (2002)

Batah and Gore (2009)

$$k_{190}:max\left(1/\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i(\max(\widehat{a}_i))^2}\right)\quad k_{191}:max\left(\sqrt{\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i(\max(\widehat{a}_i))^2}}\right)\quad k_{192}:max\left(1/\sqrt{\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i(\max(\widehat{a}_i))^2}}\right)$$

$$k_{193}:\frac{1}{p}\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i(\max(\widehat{a}_i))^2}\quad k_{194}:1/\frac{1}{p}\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i(\max(\widehat{a}_i))^2}\quad k_{195}:\sqrt{\frac{1}{p}\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i(\max(\widehat{a}_i))^2}}$$

$$k_{196}:1/\sqrt{\frac{1}{p}\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i(\max(\widehat{a}_i))^2}}\quad k_{197}:p\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i(\max(\widehat{a}_i))^2}\quad k_{198}:\frac{1}{p\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i(\max(\widehat{a}_i))^2}}$$

$$k_{199}:\sqrt{p\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i(\max(\widehat{a}_i))^2}}\quad k_{200}:1/\sqrt{p\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i(\max(\widehat{a}_i))^2}}\quad k_{201}:\left(\prod_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i(\max(\widehat{a}_i))^2}\right)^{1/p}$$

$$k_{202}:\frac{1}{\left(\prod_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i(\max(\widehat{a}_i))^2}\right)^{\frac{1}{p}}}\quad k_{203}:\sqrt{\left(\prod_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i(\max(\widehat{a}_i))^2}\right)^{\frac{1}{p}}}\quad k_{204}:1/\sqrt{\left(\prod_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i(\max(\widehat{a}_i))^2}\right)^{\frac{1}{p}}}$$

$$k_{205}:med\left(\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i(\max(\widehat{a}_i))^2}\right)\quad k_{206}:med\left(1/\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i(\max(\widehat{a}_i))^2}\right)\quad k_{207}:med\left(\sqrt{\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i(\max(\widehat{a}_i))^2}}\right)$$

$$k_{208}:med\left(1/\sqrt{\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i(\max(\widehat{a}_i))^2}}\right)\quad k_{209}:max\left(1/\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+[\max(\lambda_i)]\widehat{a}_i^2}\right)\quad k_{210}:max\left(\sqrt{\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+[\max(\lambda_i)]\widehat{a}_i^2}}\right)$$

$$k_{211}:max\left(1/\sqrt{\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+[\max(\lambda_i)]\widehat{a}_i^2}}\right)\quad k_{212}:\frac{1}{p}\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+[\max(\lambda_i)]\widehat{a}_i^2}\quad k_{213}:1/\frac{1}{p}\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+[\max(\lambda_i)]\widehat{a}_i^2}$$

$$k_{214}:\sqrt{\frac{1}{p}\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+[\max(\lambda_i)]\widehat{a}_i^2}}\quad k_{215}:1/\sqrt{\frac{1}{p}\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+[\max(\lambda_i)]\widehat{a}_i^2}}\quad k_{216}:p\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+[\max(\lambda_i)]\widehat{a}_i^2}$$

$$k_{217}:1/p\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+[\max(\lambda_i)]\widehat{a}_i^2}\quad k_{218}:\sqrt{p\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+[\max(\lambda_i)]\widehat{a}_i^2}}\quad k_{219}:1/\sqrt{p\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+[\max(\lambda_i)]\widehat{a}_i^2}}$$

$$k_{220}:\left(\prod_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+[\max(\lambda_i)]\widehat{a}_i^2}\right)^{1/p}\quad k_{221}:1/\left(\prod_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+[\max(\lambda_i)]\widehat{a}_i^2}\right)^{1/p}\quad k_{222}:\sqrt{\left(\prod_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+[\max(\lambda_i)]\widehat{a}_i^2}\right)^{1/p}}$$

$$k_{223}:1/\sqrt{\left(\prod_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+[\max(\lambda_i)]\widehat{a}_i^2}\right)^{1/p}}\quad k_{224}:med\left(\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+[\max(\lambda_i)]\widehat{a}_i^2}\right)\quad k_{225}:med\left(1/\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+[\max(\lambda_i)]\widehat{a}_i^2}\right)$$

$$k_{226}:med\left(\sqrt{\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+[\max(\lambda_i)]\widehat{a}_i^2}}\right)\quad k_{227}:med\left(1/\sqrt{\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+[\max(\lambda_i)]\widehat{a}_i^2}}\right)\quad k_{228}:max\left(1/\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{a}_i^2}\right)$$

$$k_{229}:max\left(\sqrt{\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{a}_i^2}}\right)\quad k_{230}:max\left(1/\sqrt{\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{a}_i^2}}\right)\quad k_{231}:\frac{1}{p}\sum_{i=1}^{p}\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{a}_i^2}\quad k_{232}:\frac{1}{\frac{1}{p}\sum_{i=1}^{p}\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{a}_i^2}}$$

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$$k_{233}:\sqrt{\frac{1}{p}\sum_{i=1}^{p}\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{\alpha}_i^2}}k_{234}:1/\sqrt{\frac{1}{p}\sum_{i=1}^{p}\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{\alpha}_i^2}}k_{235}:p\sum_{i=1}^{p}\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{\alpha}_i^2}k_{236}:\frac{1}{p\sum_{i=1}^{p}\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{\alpha}_i^2}}$$

$$k_{237}:\sqrt{p\sum_{i=1}^{p}\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{\alpha}_i^2}}k_{238}:1/\sqrt{p\sum_{i=1}^{p}\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{\alpha}_i^2}}k_{239}:\left(\prod_{i=1}^{p}\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{\alpha}_i^2}\right)^{\frac{1}{p}}k_{240}:\frac{1}{\left(\prod_{i=1}^{p}\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{\alpha}_i^2}\right)^{1/p}}$$

$$k_{241}:\sqrt{\left(\prod_{i=1}^{p}\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{\alpha}_i^2}\right)^{1/p}}k_{242}:\frac{1}{\sqrt{\left(\prod_{i=1}^{p}\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{\alpha}_i^2}\right)^{1/p}}}k_{243}:med\left(\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{\alpha}_i^2}\right)k_{244}:med\left(\frac{1}{\frac{max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{\alpha}_i^2}}\right)$$

$$k_{245}:med\left(\sqrt{\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{\alpha}_i^2}}\right)k_{246}:med\left(1/\sqrt{\frac{\max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{\alpha}_i^2}}\right)k_{247}:max\left(1/\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+max(\lambda_i)max(\widehat{\alpha}_i^2)}\right)$$

$$k_{248}:max\left(\sqrt{\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\max(\lambda_i)\max(\widehat{\alpha}_i^2)}}\right)k_{249}:max\left(\frac{1}{\sqrt{\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+max(\lambda_i)max(\widehat{\alpha}_i^2)}}}\right)k_{250}:\frac{1}{p}\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+max(\lambda_i)max(\widehat{\alpha}_i^2)}$$

$$k_{251}:\frac{1}{\frac{1}{p}\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\max(\lambda_i)\max(\widehat{\alpha}_i^2)}}k_{252}:\sqrt{\frac{1}{p}\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+max(\lambda_i)max(\widehat{\alpha}_i^2)}}k_{253}:\frac{1}{\sqrt{\frac{1}{p}\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+max(\lambda_i)max(\widehat{\alpha}_i^2)}}}$$

$$k_{254}:p\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\max(\lambda_i)\max(\widehat{\alpha}_i^2)}k_{255}:\frac{\mathbf{1}}{p\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\max(\lambda_i)\max(\widehat{\alpha}_i^2)}}k_{256}:\sqrt{p\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\max(\lambda_i)\max(\widehat{\alpha}_i^2)}}$$

$$k_{257}:1/\sqrt{p\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\max(\lambda_i)\max(\widehat{\alpha}_i^2)}}k_{258}:\left(\prod_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\max(\lambda_i)\max(\widehat{\alpha}_i^2)}\right)^{\frac{1}{p}}k_{259}:\frac{1}{\left(\prod_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\max(\lambda_i)\max(\widehat{\alpha}_i^2)}\right)^{\frac{1}{p}}}$$

$$k_{260}:\sqrt{\left(\prod_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+max(\lambda_i)max(\widehat{\alpha}_i^2)}\right)^{\frac{1}{p}}}k_{261}:1/\sqrt{\left(\prod_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+max(\lambda_i)max(\widehat{\alpha}_i^2)}\right)^{\frac{1}{p}}}k_{262}:med\left(\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+max(\lambda_i)max(\widehat{\alpha}_i^2)}\right)k_{263}:$$

$$med\left(\frac{1}{\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+max(\lambda_i)max(\widehat{\alpha}_i^2)}}\right)k_{264}:med\left(\sqrt{\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\max(\lambda_i)\max(\widehat{\alpha}_i^2)}}\right)$$

$$k_{265}:med\left(1/\sqrt{\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\max(\lambda_i)\max(\widehat{\alpha}_i^2)}}\right)k_{266}:max\left(1/\frac{max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}\right)k_{267}:max\left(\sqrt{\frac{max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}}\right)k_{268}:\frac{1}{p}\sum_{i=1}^{p}\frac{max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}k_{269}:$$

$$\frac{1}{\frac{1}{p}\sum_{i=1}^{p}\frac{max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}}k_{270}:\sqrt{\frac{1}{p}\sum_{i=1}^{p}\frac{max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}}$$

$$k_{271}:1/\sqrt{\frac{1}{p}\sum_{i=1}^{p}\frac{max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}}k_{272}:p\sum_{i=1}^{p}\frac{max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}k_{273}:\frac{1}{p\sum_{i=1}^{p}\frac{max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}}k_{274}:\sqrt{p\sum_{i=1}^{p}\frac{max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}}$$

$$k_{275}:1/\sqrt{p\sum_{i=1}^{p}\frac{max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}}k_{276}:\left(\prod_{i=1}^{p}\frac{max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}\right)^{\frac{1}{p}}$$

$$k_{277}:1/\left(\prod_{i=1}^{p}\frac{max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}\right)^{\frac{1}{p}}k_{278}:\sqrt{\left(\prod_{i=1}^{p}\frac{max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}\right)^{\frac{1}{p}}}k_{279}:1/\sqrt{\left(\prod_{i=1}^{p}\frac{max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}\right)^{\frac{1}{p}}}$$

$$k_{280}:med\left(\frac{max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}\right)k_{281}:med\left(1/\frac{max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}\right)k_{282}:med\left(\sqrt{\frac{max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}}\right)$$

$$k_{283}:med\left(1/\sqrt{\frac{max(\lambda_i)\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}}\right)k_{284}:max\left(1/\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}\right)k_{285}:max\left(1/\sqrt{\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}}\right)k_{286}:\frac{1}{p}\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}$$

$$k_{287}:p/\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}k_{288}:\sqrt{\frac{1}{p}\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}}k_{289}:1/\sqrt{\frac{1}{p}\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}}k_{290}:p\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}$$

$$k_{291}:1/\left(p\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}\right)k_{292}:\sqrt{p\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}}k_{293}:1/\sqrt{p\sum_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}}k_{294}:\left(\prod_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}\right)^{1/p}$$

$$k_{295}:1/\left(\prod_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}\right)^{\frac{1}{p}}k_{296}:\sqrt{\left(\prod_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}\right)^{\frac{1}{p}}}k_{297}:1/\sqrt{\left(\prod_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}\right)^{\frac{1}{p}}}k_{298}:med\left(\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}\right)k_{299}:$$

$$med\left(\frac{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}{\lambda_i\widehat{\sigma}^2}\right)k_{300}:med\left(\sqrt{\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}}\right)k_{301}:med\left(1/\sqrt{\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\max(\widehat{\alpha}_i^2)}}\right)$$ Lukman et al. (2018)

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$k_{302} : max\left(\frac{\lambda_i\widehat{\sigma}^2}{\widehat{\alpha}_i^2}\right)$ $k_{303} : \sum_{i=1}^{p}\left(\frac{\lambda_i\widehat{\sigma}^2}{\widehat{\alpha}_i^2}\right)/p$ $k_{304} : max\left(\frac{(n-p)\widehat{\sigma}^2+\lambda_{max}\widehat{\alpha}_i^2}{\lambda_{min}}\right)$ Qasim et al. (2021)

$k_{305} : \frac{p\widehat{\sigma}^2}{\widehat{\alpha}'\widehat{\alpha}}+\frac{1}{n\lambda_{max}}$ Bhat and Vidya (2014)

$k_{306} : \sqrt{\frac{p\widehat{\sigma}^2}{\widehat{\alpha}'\widehat{\alpha}}max\left(0,\frac{p\widehat{\sigma}^2}{\widehat{\alpha}'\widehat{\alpha}}-\frac{1}{n(VIF_j)_{max}}\right)}$ *where* $VIF_j=\frac{1}{(1-R_j^2)}, j=1,2,\ldots,p$ $k_{307} : \frac{2\frac{p\widehat{\sigma}^2}{\widehat{\alpha}'\widehat{\alpha}}max\left(0,\frac{p\widehat{\sigma}^2}{\widehat{\alpha}'\widehat{\alpha}}-\frac{1}{n(VIF_j)_{max}}\right)}{\frac{p\widehat{\sigma}^2}{\widehat{\alpha}'\widehat{\alpha}}+max\left(0,\frac{p\widehat{\sigma}^2}{\widehat{\alpha}'\widehat{\alpha}}-\frac{1}{n(VIF_j)_{max}}\right)}$ Bhat and Vidya (2017)

$k_{308} : \frac{w_2\frac{p\widehat{\sigma}^2}{\widehat{\alpha}'\widehat{\alpha}}+w_1\ max\left(0,\frac{p\widehat{\sigma}^2}{\widehat{\alpha}'\widehat{\alpha}}-\frac{1}{n(VIF_j)_{max}}\right)}{w_1+w_2}$ where w_1 and w_2 are the largest and second-largest eigenvalues. Bhat (2019)

$k_{309} : exp\left(\frac{w_2\ln\left(\frac{p\widehat{\sigma}^2}{\widehat{\alpha}'\widehat{\alpha}}\right)+w_1\ln\left(max\left(0,\frac{p\widehat{\sigma}^2}{\widehat{\alpha}'\widehat{\alpha}}-\frac{1}{n(VIF_j)_{max}}\right)\right)}{w_1+w_2}\right)$ $k_{310} : \frac{w_1+w_2}{\left(w_2/\frac{p\widehat{\sigma}^2}{\widehat{\alpha}'\widehat{\alpha}}\right)+\left(w_1/max\left(0,\frac{p\widehat{\sigma}^2}{\widehat{\alpha}'\widehat{\alpha}}-\frac{1}{n(VIF_j)_{max}}\right)\right)}$ Bhat and Vidya (2020)

$k_{311} : \frac{\lambda_{med}\sum_{i=1}^{p}\widehat{\alpha}_i^2}{\lambda_{max}}$ $k_{312} : \left|\frac{p\widehat{\sigma}^2}{\widehat{\alpha}'\widehat{\alpha}}-\frac{p\widehat{\sigma}^2}{\widehat{\alpha}'X'X\widehat{\alpha}}\right|$ $k_{313} : min\left(\sqrt{\frac{\lambda_{min}\sum_{i=1}^{p}\widehat{\alpha}_i^2}{\widehat{\sigma}^2}}\right)$ $k_{314} : max\left(\frac{\lambda_{min}\sum_{i=1}^{p}\widehat{\alpha}_i^2}{\sqrt{\widehat{\sigma}^2}}\right)$ Lattef and Alheety (2020)

$k_{315} : \frac{\widehat{\sigma}^2}{\widehat{\alpha}_{max}^2}\frac{\lambda_{max}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{316} : \frac{\widehat{\sigma}^2}{\left(\prod_{i=1}^{p}\widehat{\alpha}_i^2\right)^{1/p}}\frac{\lambda_{max}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{317} : med\left(\frac{\widehat{\sigma}^2}{\widehat{\alpha}_i^2}\right)\frac{\lambda_{max}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{318} : \frac{\lambda_{max}\widehat{\sigma}^2}{(n-p-1)\widehat{\sigma}^2+\lambda_{max}\widehat{\alpha}_{max}^2}\frac{\lambda_{max}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{319} :$

$max\left(\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{\alpha}_i^2}\right)\frac{\lambda_{max}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{320} : med\left(\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{\alpha}_i^2}\right)\frac{\lambda_{max}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{321} : \left(\prod_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{\alpha}_i^2}\right)^{1/p}\frac{\lambda_{max}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$

$k_{322} : max(1/\sqrt{\widehat{\sigma}^2/\widehat{\alpha}_i^2})\frac{\lambda_{max}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{323} : \left(\prod_{i=1}^{p}1/\sqrt{\widehat{\sigma}^2/\widehat{\alpha}_i^2}\right)^{1/p}\frac{\lambda_{max}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{324} : \left(\prod_{i=1}^{p}\sqrt{\frac{\widehat{\sigma}^2}{\widehat{\alpha}_i^2}}\right)^{1/p}\frac{\lambda_{max}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{325} : med(1/\sqrt{\widehat{\sigma}^2/\widehat{\alpha}_i^2})\frac{\lambda_{max}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{326} : max\left(1/\sqrt{\frac{(\lambda_{max}\widehat{\sigma}^2)}{((n-p)\widehat{\sigma}^2+\lambda_{max}\widehat{\alpha}_i^2)}}\right)\frac{\lambda_{max}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$

$k_{327} : max\left(\sqrt{\frac{(\lambda_{max}\widehat{\sigma}^2)}{((n-p)\widehat{\sigma}^2+\lambda_{max}\widehat{\alpha}_i^2)}}\right)\frac{\lambda_{max}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{328} : \left(\prod_{i=1}^{p}1/\sqrt{\frac{(\lambda_{max}\widehat{\sigma}^2)}{((n-p)\widehat{\sigma}^2+\lambda_{max}\widehat{\alpha}_i^2)}}\right)^{1/p}\frac{\lambda_{max}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{329} :$

$\left(\prod_{i=1}^{p}\sqrt{\frac{(\lambda_{max}\widehat{\sigma}^2)}{((n-p)\widehat{\sigma}^2+\lambda_{max}\widehat{\alpha}_i^2)}}\right)^{1/p}\frac{\lambda_{max}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{330} : med\left(\frac{1}{\sqrt{\frac{(\lambda_{max}\widehat{\sigma}^2)}{((n-p)\widehat{\sigma}^2+\lambda_{max}\widehat{\alpha}_i^2)}}}\right)\frac{\lambda_{max}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ Khalaf, Mansson, and Shukur (2013)

$k_{gm}=\frac{\widehat{\sigma}^2}{\left(\prod_{i=1}^{p}\widehat{\alpha}_i^2\right)^{1/p}}k_{med}=med\left(\frac{\widehat{\sigma}^2}{\widehat{\alpha}_i^2}\right)k_{km3}=max\left(\sqrt{\frac{\widehat{\sigma}^2}{\widehat{\alpha}_i^2}}\right)\quad k_{hmo}=p\widehat{\sigma}^2/\sum_{i=1}^{p}\left\{\widehat{\alpha}_i^2/\left[1+\left(1+\lambda_i\left(\frac{\widehat{\alpha}_i^2}{\widehat{\sigma}^2}\right)^{1/2}\right)\right]\right\}$

$$k_{cjh}=\begin{cases}\frac{p\widehat{\sigma}^2}{(\widehat{\beta}_{OLS}-J)'(\widehat{\beta}_{OLS}-J)} & if\ (\widehat{\beta}_{OLS}-J)'(\widehat{\beta}_{OLS}-J)>\widehat{\sigma}^2tr(X'X)^{-1}\\ \frac{p\widehat{\sigma}^2}{(\widehat{\beta}_{OLS}-J)'(\widehat{\beta}_{OLS}-J)-\widehat{\sigma}^2tr(X'X)} & otherwise\end{cases}\quad \text{where } J\sim\left(\beta,\frac{\sigma^2}{k}\right)$$

$k_{fg}=\frac{p\widehat{\sigma}^2}{\sum_{i=1}^{p}[\widehat{\alpha}_i^2/\{(\widehat{\alpha}_i^4\lambda_i^2/4\widehat{\sigma}^4+6\widehat{\alpha}_i^2\lambda_i/\widehat{\sigma}^2)^{1/2}-\widehat{\alpha}_i^2\lambda_i/2\widehat{\sigma}^2\}]}$

$k_{331} : mean(k_{gm},k_{med},k_{km3},k_{hmo},k_{cjh},k_{fg})$ $k_{332} : med(k_{gm},k_{med},k_{km3},k_{hmo},k_{cjh},k_{fg})$ $k_{333} : max(k_{gm},k_{med},k_{km3},k_{hmo},k_{cjh},k_{fg})$ $k_{334} : harm.mean(k_{gm},k_{med},k_{km3},k_{hmo},k_{cjh},k_{fg})$

Kibria and Banik (2016)

$k_{335} : \frac{\widehat{\sigma}^2}{\widehat{\alpha}_{max}^2}\frac{\lambda_{geo.mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{336} : \frac{\widehat{\sigma}^2}{\left(\prod_{i=1}^{p}\widehat{\alpha}_i^2\right)^{1/p}}\frac{\lambda_{geo.mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{337} : med\left(\frac{\widehat{\sigma}^2}{\widehat{\alpha}_i^2}\right)\frac{\lambda_{geo.mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{338} : \frac{\lambda_{max}\widehat{\sigma}^2}{(n-p-1)\widehat{\sigma}^2+\lambda_{max}\widehat{\alpha}_{max}^2}\frac{\lambda_{geo.mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{339} :$

$max\left(\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{\alpha}_i^2}\right)\frac{\lambda_{geo.mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{340} : med\left(\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{\alpha}_i^2}\right)\frac{\lambda_{geo.mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{341} : \left(\prod_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{\alpha}_i^2}\right)^{1/p}\frac{\lambda_{geo.mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$

$k_{342} : max(1/\sqrt{\widehat{\sigma}^2/\widehat{\alpha}_i^2})\frac{\lambda_{geo.mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{343} : \left(\prod_{i=1}^{p}1/\sqrt{\widehat{\sigma}^2/\widehat{\alpha}_i^2}\right)^{1/p}\frac{\lambda_{geo.mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{344} : \left(\prod_{i=1}^{p}\sqrt{\frac{\widehat{\sigma}^2}{\widehat{\alpha}_i^2}}\right)^{1/p}\frac{\lambda_{geo.mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{345} : med(1/\sqrt{\widehat{\sigma}^2/\widehat{\alpha}_i^2})\frac{\lambda_{geo.mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{346} : max\left(1/\sqrt{\frac{(\lambda_{max}\widehat{\sigma}^2)}{((n-p)\widehat{\sigma}^2+\lambda_{max}\widehat{\alpha}_i^2)}}\right)\frac{\lambda_{geo.mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$

$k_{347} : max\left(\sqrt{\frac{(\lambda_{max}\widehat{\sigma}^2)}{((n-p)\widehat{\sigma}^2+\lambda_{max}\widehat{\alpha}_i^2)}}\right)\frac{\lambda_{geo.mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{348} : \left(\prod_{i=1}^{p}1/\sqrt{\frac{(\lambda_{max}\widehat{\sigma}^2)}{((n-p)\widehat{\sigma}^2+\lambda_{max}\widehat{\alpha}_i^2)}}\right)^{1/p}\frac{\lambda_{geo.mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{349} : \left(\prod_{i=1}^{p}\sqrt{\frac{(\lambda_{max}\widehat{\sigma}^2)}{((n-p)\widehat{\sigma}^2+\lambda_{max}\widehat{\alpha}_i^2)}}\right)^{1/p}\frac{\lambda_{geo.mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$

$k_{350} : med\left(1/\sqrt{\frac{(\lambda_{max}\widehat{\sigma}^2)}{((n-p)\widehat{\sigma}^2+\lambda_{max}\widehat{\alpha}_i^2)}}\right)\frac{\lambda_{geo.mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$

Suhail, Ilyas, and Ullah (2019)

$k_{351} : \frac{\widehat{\sigma}^2}{\widehat{\alpha}_{max}^2}\frac{\lambda_{mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{352} : \frac{\widehat{\sigma}^2}{\left(\prod i=1^{p}\widehat{\alpha}_i^2\right)^{1/p}}\frac{\lambda_{mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{353} : med\left(\frac{\widehat{\sigma}^2}{\widehat{\alpha}_i^2}\right)\frac{\lambda_{mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{354} : \frac{\lambda_{max}\widehat{\sigma}^2}{(n-p-1)\widehat{\sigma}^2+\lambda_{max}\widehat{\alpha}_{max}^2}\frac{\lambda_{mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{355} :$

$max\left(\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{\alpha}_i^2}\right)\frac{\lambda_{mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{356} : med\left(\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{\alpha}_i^2}\right)\frac{\lambda_{mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{357} : \left(\prod_{i=1}^{p}\frac{\lambda_i\widehat{\sigma}^2}{(n-p)\widehat{\sigma}^2+\lambda_i\widehat{\alpha}_i^2}\right)^{1/p}\frac{\lambda_{mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$

$k_{358} : max(1/\sqrt{\widehat{\sigma}^2/\widehat{\alpha}_i^2})\frac{\lambda_{mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{359} : \left(\prod_{i=1}^{p}1/\sqrt{\widehat{\sigma}^2/\widehat{\alpha}_i^2}\right)^{1/p}\frac{\lambda_{mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{360} : \left(\prod_{i=1}^{p}\sqrt{\frac{\widehat{\sigma}^2}{\widehat{\alpha}_i^2}}\right)^{1/p}\frac{\lambda_{mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{361} : med(1/\sqrt{\widehat{\sigma}^2/\widehat{\alpha}_i^2})\frac{\lambda_{mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ $k_{362} : max\left(1/\sqrt{\frac{(\lambda_{max}\widehat{\sigma}^2)}{((n-p)\widehat{\sigma}^2+\lambda_{max}\widehat{\alpha}_i^2)}}\right)\frac{\lambda_{mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$

(*continued on next page*)

(*continued*)

k_{363} : $max\left(\sqrt{\frac{(\lambda_{max}\widehat{\sigma}^2)}{((n-p)\widehat{\sigma}^2+\lambda_{max}\widehat{\alpha}_i^2)}}\right)\frac{\lambda_{mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}k_{364}$: $\left(\prod_{i=1}^{p}1/\sqrt{\frac{(\lambda_{max}\widehat{\sigma}^2)}{((n-p)\widehat{\sigma}^2+\lambda_{max}\widehat{\alpha}_i^2)}}\right)^{1/p}\frac{\lambda_{mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$

k_{365} : $\left(\prod_{i=1}^{p}\sqrt{\frac{(\lambda_{max}\widehat{\sigma}^2)}{((n-p)\widehat{\sigma}^2+\lambda_{max}\widehat{\alpha}_i^2)}}\right)^{1/p}\frac{\lambda_{mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}k_{366}$: $med\left(1/\sqrt{\frac{(\lambda_{max}\widehat{\sigma}^2)}{((n-p)\widehat{\sigma}^2+\lambda_{max}\widehat{\alpha}_i^2)}}\right)\frac{\lambda_{mean}}{\sum_{i=1}^{p}|\widehat{\alpha}|_i}$ Ullah, Suhail, and Ilyas (2017)

in terms of p, n, ρ, and σ parameters.

Considering the performance of the proposed k_{robust} estimator, the percentage of having a minimum AMSE value in 6050 cases was found to be 54.79%. This is actually increased to 96.28% in all cases with ANOVA tests included, indicating significant success. The other estimators with the highest percentage of having the minimum AMSE value after k_{robust}, were k_{149} (3.29%), k_{219} (2.89%), k_{211} (2.21%), and k_{209} (2.18%), proving that k_{robust} is superior. Even with ANOVA tests, none of the estimators have reached 30% percent level of minimum AMSE. In fact, k_{211} from these estimators had the highest percentage of having the minimum AMSE value for all cases with 28.91%. Furthermore, upon examination of the outcomes regarding the criteria of outlier detection and normality for these estimators, they were found to have insufficient performance in at least one of them. Therefore, although these estimators have minimum AMSE value after k_{robust}, they are not really promising at all.

As a result of comparing the performances of k_{robust} and 366 estimators for all cases in terms of the MSE criterion based on rank numbers, k_{robust} was found to be the most effective estimator with minimum total rank numbers value among all estimators. In addition to the MSE criterion, the outlier generation percentage of k_{robust} is as low as 0.5% and normality is assured to a strong level of 99.82%, indicating that k_{robust} has performed pretty great in terms of these two criteria as well. In conclusion, it can be stated that the developed k_{robust} ridge parameter estimator outperformed the other 366 estimators in terms of outlier detection and normality criteria as well as MSE criterion in a total of 6050 cases taken in small, medium, and large sizes, with regard to the parameters p, n, ρ, and σ.

The other findings of the study are summarized as follows: It was determined that 58 out of 366 ridge parameter estimators had an outlier generation percentage above 5%, whereas 109 were below 1%. In the studies where k_{84}, k_{144}, k_{70}, k_{162}, k_{88}, k_{164}, k_{22}, k_{163}, k_{86}, k_{73}, k_{166}, and k_8 were among the top 20 estimators with the highest outlier generation percentage, a conclusion that would reveal the superiority of these estimators was not achieved. Moreover, the ridge parameter estimators k_{16}, k_{39}, k_{11}, k_{145}, k_{147}, k_{90}, k_{302} and k_{303} were found to be more effective in some cases than the other estimators (partially or entirely) considered in the studies where they were proposed. However, we do not recommend all these estimators to be used for ridge parameter estimation due to generation of a high percentage level of outliers. When the performance of ridge parameter estimators is evaluated in terms of normality criterion, the estimators k_8, k_{16}, k_{70}, k_{84}, k_{88}, k_{144}, k_{162}, k_{164}, k_{302}, k_{303} and k_{331} were found to be violating normality in every single case. Besides these estimators, the number of cases in which 60 of 366 ridge parameter estimators are normally distributed within all cases (6050) did not exceed 3025 cases, while 185 of the remaining estimators performed well by providing normality in almost all cases. The estimators assuring normality in all cases were determined as k_{20}, k_{29}, k_{47}, k_{56}, k_{57}, k_{58}, k_{60}, k_{62}, k_{107}, k_{109}, k_{111}, k_{113}, k_{115}, k_{117}, k_{120}, k_{122}, k_{124}, k_{126}, k_{127}, k_{129}, k_{131}, k_{133}, k_{139}, k_{214}, k_{222}, k_{226}, k_{229}, k_{231}, k_{233}, k_{234}, k_{235}, k_{237}, k_{239}, k_{241}, k_{243}, k_{245}, k_{267}, k_{268}, k_{270}, k_{276}, k_{278}, k_{280}, k_{290} and k_{292}.

Finally, according to the results of Spearman's rank correlation analysis, among total rank numbers, outlier generation ratios, and normality ratios obtained for the estimators, as the outlier generation ratios of the estimators decrease, the normality ratios tend to increase. Thus, their performance in terms of the MSE criterion increases significantly as well, depending on the decrease in the sum of rank numbers. In addition, it was determined that there was a highly significant inverse correlation between outlier generation and normality criteria.

As a result, the k_{robust} estimator, which does not generate outliers, provides both the minimum MSE value and assures normality, regardless of the sample size, degree of multicollinearity, and number of regressors. In this way, the prevalent confusion in the literature on determining the optimal ridge parameter estimator in ridge regression has been avoided. In the mean time, the new search method can be enhanced by considering a new generation process allowing multilinear relationships among regressors. We suggest readers working on developing ridge regression parameters to consider a new generation process method for future work.

Acknowledgement

This study was supported by TÜBİTAK (The Scientific and Technological Research Council of Türkiye) with project number 121F257. We express our gratitude to TÜBİTAK for their support of our study.

Appendix

References

Al-Hassan, Y. M. (2010). Performance of a new ridge regression estimator. *Journal of the Association of Arab Universities for Basic and Applied Sciences, 9*, 23–26. https://doi.org/10.1016/j.jaubas.2010.12.006

Alkhamisi, M. A., Khalaf, G., & Shukur, G. (2006). Some modifications for choosing ridge parameters. *Communications in Statistics - Theory and Methods, 35*, 2005–2020. https://doi.org/10.1080/03610920600762905

Alkhamisi, M. A., & Shukur, G. (2007). A Monte Carlo study of recent ridge parameters. *Communication in Statistics - Simulation and Computation, 36*, 535–547. https://doi.org/10.1080/03610910701208619

Alkhamisi, M. A., & Shukur, G. (2008). Developing ridge parameters for SUR model. *Communications in Statistics - Theory and Methods, 37*, 544–564. https://doi.org/10.1080/03610920701469152

Armstrong, R. A. (2019). Is there a large sample size problem? *Ophthalmic and Physiological Optics, 39*, 129–130. https://doi.org/10.1111/opo.12618

Asar, Y., & Genç, A. (2017). A note on some new modifications of ridge estimators. *Kuwait Journal of Science, 44*, 75–82.

Asar, Y., Karaibrahimoğlu, A., & Genç, A. (2014). Modified ridge regression parameters: A comparative Monte Carlo study. *Hacettepe Journal of Mathematics and Statistics, 43*, 827–841.

Batah, F. S. M., & Gore, S. D. (2009). Ridge regression estimator: Combining unbiased and ordinary ridge regression methods of estimation. *Surveys in Mathematics and Its Applications, 4*, 99–109.

Bhat, S. (2019). Performance of a weighted ridge estimator. *International Journal of Agricultural and Statistical Sciences, 15*, 347–354.
Bhat, S., & Vidya, R. (2014). Improvement on ridge regression estimator. In *Proceedings of National Conference on statistical methods and data analysis*. Mysore, India.
Bhat, S., & Vidya, R. (2016). A comparative study on the performance of new ridge estimators. *Pakistan Journal of Statistics and Operation Research, 12*, 317–325. https://doi.org/10.18187/pjsor.v12i2.1188
Bhat, S., & Vidya, R. (2017). A new ridge estimator and its performance. In *Proceedings of National Conference on operations Management, Analytics and statistical methods*. Mysore, India.
Bhat, S., & Vidya, R. (2020). Performance of ridge estimators based on weighted geometric mean and harmonic mean. *Journal of Scientific Research, 12*, 1–13. https://doi.org/10.3329/jsr.v12i1.40525
Dorugade, A. V. (2014a). New ridge parameters for ridge regression. *Journal of the Association of Arab Universities for Basic and Applied Sciences, 15*, 94–99. https://doi.org/10.1016/j.jaubas.2013.03.005
Dorugade, A. V. (2014b). On comparison of some ridge parameters in ridge regression. *Sri Lankan Journal of Applied Statistics, 15*, 31–45. https://doi.org/10.4038/sljastats.v15i1.6792
Dorugade, A. V. (2016). Improved ridge estimator in linear regression with multicollinearity, heteroscedastic errors and outliers. *Journal of Modern Applied Statistical Methods, 15*, 362–381. https://doi.org/10.22237/jmasm/1478002860
Dorugade, A. V., & Kashid, D. N. (2010). Alternative method for choosing ridge parameter for regression. *International Journal of Applied Mathematical Sciences, 4*, 447–456.
Fahrmeir, L., Kneib, T., Lang, S., & Marx, B. D. (2021). *Regression: Models, methods and applications* (2nd ed.). Germany: Springer-Verlag GmbH. https://doi.org/10.1007/978-3-642-34333-9
Frisch, R. (1934). *Statistical confluence analysis by means of complete regression systems (Publication 5)*. Oslo: University Institute of Economics.
Frost, J. (2019). *Regression analysis: An intuitive guide using and interpreting linear models*. United States: Statistics By Jim Publishing.
Gibbons, D. G. (1981). A simulation study of some ridge estimators. *Journal of the American Statistical Association, 76*, 131–139. https://doi.org/10.2307/2287058
Göktaş, A., Akkuş, Ö., & Kuvat, A. (2021). A new robust ridge parameter estimator based on search method for linear regression model. *Journal of Applied Statistics, 48*, 2457–2472. https://doi.org/10.1080/02664763.2020.1803814
Göktaş, A., & Sevinç, V. (2016). Two new ridge parameters and a guide for selecting an appropriate ridge parameter in linear regression. *Gazi University Journal of Science, 29*, 201–211.
Haggag, M. M. M. (2016). A new optimal selection of a ridge regression parameter. *International Journal of Applied Mathematics & Statistical Sciences, 5*, 101–132.
Hocking, R. R., Speed, F. M., & Lynn, M. J. (1976). A class of biased estimators in linear regression. *Technometrics, 18*, 425–437. https://doi.org/10.2307/1268658
Hoerl, A. E., & Kennard, R. W. (1970). Ridge regression: Biased estimaton to nonorthogonal problems. *Technometrics, 12*, 69–82. https://doi.org/10.1080/00401706.1970.10488634
Hoerl, A. E., Kennard, R. W., & Baldwin, K. F. (1975). Ridge regression: Some simulations. *Communications in Statistics, 4*, 105–123. https://doi.org/10.1080/03610927508827232
Hoffmann, J. P. (2022). *Linear regression models: Applications in R. United States*. Chapman and Hall/CRC Press. https://doi.org/10.1201/9781003162230
Karaibrahimoğlu, A., Asar, Y., & Genç, A. (2016). Some new modifications of Kibria's and Dorugade's methods: An application to Turkish GDP data. Journal of the Association of Arab Universities for Basic and Applied Sciences, 20, 89-99. https://doi.org/10.1016/j.jaubas.2014.08.005.
Khalaf, G. (2012). A proposed ridge parameter to improve the least squares estimator. *Journal of Modern Applied Statistical Methods, 11*, 443–449. https://doi.org/10.22237/jmasm/1351743240
Khalaf, G. (2013). A comparison between biased and unbiased estimators in ordinary least squares regression. *Journal of Modern Applied Statistical Methods, 12*, 293–303. https://doi.org/10.22237/jmasm/1383279360
Khalaf, G., & Iguernane, M. (2014). Ridge regression and I11-conditioning. *Journal of Modern Applied Statistical Methods, 13*, 355–363. https://doi.org/10.22237/jmasm/1414815420
Khalaf, G., & Iguernane, M. (2016). Multicollinearity and a ridge parameter estimation approach. *Journal of Modern Applied Statistical Methods, 15*, 400–410. https://doi.org/10.22237/jmasm/1478002980
Khalaf, G., Mansson, K., & Shukur, G. (2013). Modified ridge regression estimators. *Communications in Statistics - Theory and Methods, 42*, 1476–1487. https://doi.org/10.1080/03610926.2011.593285
Khalaf, G., & Shukur, G. (2005). Choosing ridge parameter for regression problems. *Communications in Statistics - Theory and Methods, 34*, 1177–1182. https://doi.org/10.1081/STA-200056836
Kibria, B. M. G. (2003). Performance of some new ridge regression estimators. *Communications in Statistics - Simulation and Computation, 32*, 419–435. https://doi.org/10.1081/SAC-120017499
Kibria, B. M. G. (2022). More than hundred (100) estimators for estimating the shrinkage parameter in a linear and generalized linear ridge regression models. *Journal of Econometrics and Statistics, 2*, 233–252. https://doi.org/10.47509/JES.2022.v02i02.06
Kibria, B. M. G., & Banik, S. (2016). Some ridge regression estimators and their performances. *Journal of Modern Applied Statistical Methods, 15*, 206–238. https://doi.org/10.22237/jmasm/1462075860
Kibria, B. M. G., & Lukman, A. F. (2020). A new ridge-type estimator for the linear regression model: Simulations and applications. *Scientifica, 2020*. https://doi.org/10.1155/2020/9758378
Kibria, B. M. G., Mansson, K., & Shukur, G. (2012). Performance of some logistic ridge regression estimators. *Computational Economics, 40*, 401–414. https://doi.org/10.1007/s10614-011-9275-x
Kim, H. Y. (2013). Statistical notes for clinical researchers: Assessing normal distribution (2) using skewness and kurtosis. *Restorative Dentistry & Endodontics, 38*, 52–54. https://doi.org/10.5395/rde.2013.38.1.52
Lattef, M. N., & Alheety, M. I. (2020). Study of some kinds of ridge regression estimators in linear regression model. *Tikrit Journal of Pure Science, 25*, 130–142. https://doi.org/10.25130/tjps.v25i5.301
Lawless, J. F., & Wang, P. (1976). A simulation study of ridge and other regression estimators. *Communications in Statistics - Theory and Methods, 5*, 307–323. https://doi.org/10.1080/03610927608827353
Leys, C., Ley, C., Klein, O., Bernard, P., & Licata, L. (2013). Detecting outliers: Do not use standard deviation around the mean, use absolute deviation around the median. *Journal of Experimental Social Psychology, 49*, 764–766. https://doi.org/10.1016/j.jesp.2013.03.013
Lukman, A. F., & Ayinde, K. (2017). Review and classifications of the ridge parameter estimation techniques. *Hacettepe Journal of Mathematics and Statistics, 46*, 953–968. https://doi.org/10.15672/hjms.201815671
Lukman, A. F., Ayinde, K., & Ajiboye, A. S. (2017). Monte Carlo study of some classification-based ridge parameter estimators. *Journal of Modern Applied Statistical Methods, 16*, 428–451. https://doi.org/10.22237/jmasm/1493598240
Lukman, A. F., & Olatunji, A. (2018). Newly proposed estimator for ridge parameter: An application to the Nigerian economy. *Pakistan Journal of Statistics, 34*, 91–98.
Lukman, A. F., Oluyemi, O. A., Akanbi, O. B., & Clement, O. A. (2018). Classification-based ridge estimation techniques of Alkhamisi methods. *Journal of Probability and Statistical Science, 16*, 165–181.
Mermi, S., Göktaş, A., & Akkuş, Ö. (2021). Are most proposed ridge parameter estimators skewed and do they have any effect on MSE values? *Journal of Statistical Computation and Simulation, 91*, 2074–2093. https://doi.org/10.1080/00949655.2021.1883611
Muniz, G., & Kibria, B. M. G. (2009). On some ridge regression estimators: An empirical comparisons. *Communications in Statistics - Simulation and Computation, 38*, 621–630. https://doi.org/10.1080/03610910802592838
Muniz, G., Kibria, B. M. G., Mansson, K., & Shukur, G. (2012). On developing ridge regression parameters: A graphical investigation. *Statistics and Operations Research Transactions, 36*, 115–138.
Nomura, M. (1988). On the almost unbiased ridge regression estimator. *Communications in Statistics - Simulation and Computation, 17*, 729–743. https://doi.org/10.1080/03610918808812690
Qasim, M., Mansson, K., & Kibria, B. M. G. (2021). On some beta ridge regression estimators: Method, simulation and application. *Journal of Statistical Computation and Simulation, 91*, 1699–1712. https://doi.org/10.1080/00949655.2020.1867549
Royall, R. M. (1986). The effect of sample size on the meaning of significance tests. *The American Statistician, 40*, 313–315. https://doi.org/10.2307/2684616
Saleh, A. K. M. E., Arashi, M., & Kibria, B. M. G. (2019). *Theory of ridge regression estimation with applications*. New Jersey: John Wiley & Sons. https://doi.org/10.1002/9781118644478
Suhail, M., Ilyas, M., & Ullah, A. (2019). Modified Ridge regression estimators: A simulation study. In *Proceedings of 15th Islamic Countries Conference on statistical Sciences*. Lahore.
Thisted, R. A. (1976). *Ridge regression, minimax estimation, and empirical bayes methods*. Stanford University.
Ullah, A., Suhail, M., & Ilyas, M. (2017). Modified method for choosing ridge parameter. *Journal of Statistics, 24*, 20–34.
Venables, W. N., & Ripley, B. D. (2002). *Modern applied statistics with S*. New York: Springer Science+Business Media, LLC. https://doi.org/10.1007/978-0-387-21706-2
West, S. G., Finch, J. F., & Curran, P. J. (1995). Structural equation models with nonnormal variables: Problems and remedies. In R. H. Hoyle (Ed.), *Structural equation modeling: Concepts, issues, and applications* (pp. 56–75). Sage Publications Inc.