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ORIGINAL RESEARCH PAPER

Some Caristi type fixed point theorems

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Abstract

In 1976, Caristi fixed point theorem was proved for ordinary metric spaces. After that, it was shown that this result is equivalent to Ekeland variational principle which has a great number of applications in many branches of mathematics. In this paper, we first introduce a new concept so called strong M_b-metric to remedy lackness of continuity of M_b-metric. Then, we investigate whether Caristi fixed point theorem can be extended to this space. Next, we obtain Caristi type fixed point theorem and some generalizations on strong M_b-metric spaces. Also, we provide some illustrative and interesting examples showing that our theorems extend the results existing in the literature. Finally, we present some applications of our results to ordinary metric spaces.

Keywords Strong M_b-metric space · Fixed point · Caristi type

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1 Introduction and preliminaries

Let (S, ρ) be a metric space and $\varphi : S \to [0, \infty)$ be a lower semicontinuous function. If the mapping $T : S \to S$ satisfies

$$\rho(\zeta, T\zeta) \leq \varphi(\zeta) - \varphi(T\zeta) \tag{1.1}$$

for all $\zeta \in S$, then it is said to be Caristi mapping. Caristi showed that every Caristi mapping on a complete metric space has a fixed point in this space [8]. This result is known as Caristi fixed point theorem in the literature. Besides, Ekeland obtained a result for a variational principle which has many applications in mathematics [13]. Then, it was shown that Caristi fixed point theorem is equivalent to Ekeland variational principle [13, 22]. So, by virtue of its applicability, Caristi fixed point theorem has been studied by many authors on some abstract spaces; see, for example, [1–3, 6, 18–20, 23] and references therein.

On the other hand, Czerwik defined a notion of b-metric to generalize the concept of standard metric [10]. Recently, many authors have studied fixed point theory on b-metric spaces [5, 9, 11, 14, 16]. However, b-metric has a disadvantage since it may not be continuous; see, example 3.9 and example 3.10 in [4]. To remedy the lackness of continuity of the b-metric, Kirk et al. introduced a notion so called strong b-metric and then, they obtained Caristi fixed point theorem on strong b-metric spaces [15]. Afterwards, they asked whether Caristi fixed point theorem can be extended to b-metric spaces. However, Dung et al. gave a negative answer to this question; see example 2.8 in [12]. Now, we recall the notion of strong b-metric space and Caristi fixed point theorem on this space:

Definition 1 [15] Let S be a nonempty set, $s \geq 1$ and $\rho : S \times S \to [0, \infty)$ be a mapping such that

$m1)$ $\rho(\zeta, \eta) = 0 \iff \zeta = \eta$,
$m2)$ $\rho(\zeta, \eta) = \rho(\eta, \zeta)$,
$m3)$ $\rho(\zeta, \eta) \leq s\rho(\zeta, \xi) + \rho(\eta, \xi)$,

for all $\zeta, \eta, \xi \in S$. Then ρ is called strong b-metric on S with the coefficient $s \geq 1$. Also, (S, ρ, s) is called a strong b-metric space.

Theorem 1 [15] *Let (S, ρ, s) be a complete strong b-metric space and $\varphi : S \to [0, \infty)$ be a lower semicontinuous function. If the mapping $T : S \to S$ satisfies inequality (1.1), then T has a fixed point.*

Very recently, Sahin et al. [21] modified the definition of M_b -metric introduced by Nabil et al. [17] to be more suitable to generalize ordinary metric, b-metric and M-metric as follows:

Definition 2 [21] Let S be a nonempty set, $s \geq 1$ and $m_b : S \times S \to [0, \infty)$ be a mapping such that

$m1)$ $m_b(\zeta, \eta) = m_b(\zeta, \zeta) = m_b(\eta, \eta) \Leftrightarrow \zeta = \eta$,
$m2)$ $m_{b\zeta\eta} = \min\{m_b(\zeta, \zeta), m_b(\eta, \eta)\} \leq m_b(\zeta, \eta)$,

$m3)$ $m_b(\zeta,\eta) = m_b(\eta,\zeta)$,
$m4)$ $m_b(\zeta,\eta) - m_{b\zeta\eta} \leq s\left[(m_b(\zeta,\xi) - m_{b\zeta\xi}) + \left(m_b(\eta,\xi) - m_{b\eta\xi}\right)\right]$.

for all $\zeta,\eta,\xi \in S$. Then it is called an m_b-metric on S with the coefficient $s \geq 1$ and (S, m_b, s) is called an M_b-metric space.

As we noticed before, although Caristi fixed point theorem can not be proved on b-metric spaces [12], this result was obtained on strong b -metric spaces [15]. In this paper, we first introduced a concept of strong M_b-metric to eliminate the discontinuity problem of M_b -metric and then we investigate whether the relation between b-metric and strong b-metric for the Caristi fixed point theorem is valid between M_b-metric and strong M_b-metric. After that, we show that Caristi's result doesn't extend to neither M_b-metric spaces nor strong M_b -metric spaces as can be seen in Example 3. Therefore, to obtain a Caristi type result on strong M_b-metric spaces, we suggest a new mapping so called Caristi mapping of new type. Thus, we present some Caristi type fixed point theorems on the strong M_b-metric spaces which are extension of Bae [6] and Suzuki [23]'s results. Also, we provide some illustrative and interesting examples showing that our theorems extend the results existing in the literature.

2 Main result

Let start to this section with definition of a strong M_b-metric space and some its properties:

Definition 3 Let S be a nonempty set, $s \geq 1$ and $m_b : S \times S \to [0,\infty)$ be a mapping such that

$sm1)$ $m_b(\zeta,\eta) = m_b(\zeta,\zeta) = m_b(\eta,\eta) \Leftrightarrow \zeta = \eta$,
$sm2)$ $m_{b\zeta\eta} = \min\{m_b(\zeta,\zeta), m_b(\eta,\eta)\} \leq m_b(\zeta,\eta)$,
$sm3)$ $m_b(\zeta,\eta) = m_b(\eta,\zeta)$,
$sm4)$ $m_b(\zeta,\eta) - m_{b\zeta\eta} \leq s(m_b(\zeta,\xi) - m_{b\zeta\xi}) + m_b(\eta,\xi) - m_{b\eta\xi}$.

for all $\zeta,\eta,\xi \in S$. Then it is called strong M_b-metric with the coefficient $s \geq 1$. Also, (S, m_b, s) is called a strong M_b-metric space.

Now, we give an example of a strong M_b-metric space:

Example 1 Let $\mathbb{N}$ be a set of all natural numbers and define the mapping $m_b : \mathbb{N} \times \mathbb{N} \to [0,\infty)$ by

$$m_b(\zeta,\eta)=\begin{cases}0, & \zeta=\eta\in\mathbb{N}\backslash\{2\}\\ 4, & \zeta=\eta=2\\ 0, & (\zeta=2\text{and}\eta=4)\text{or }(\zeta=4\text{and }\eta=2)\\ 3, & (\zeta=1\text{and}\eta=3)\text{or}(\zeta=3\text{ and }\eta=1)\\ 1, & \text{otherwise}\end{cases}.$$

In this case, it can be easily seen that $(\mathbb{N}, m_b, s)$ is a strong M_b-metric space with the coefficient $s=2$.

It is clear from the Definition 3 that every strong M_b-metric space is an M_b-metric space, but conversely may not always true. The following example shows this fact.

Example 2 Let $S=\{0,1\}\cup\left\{\frac{1}{n}:n>1,n\in\mathbb{N}\right\}$ and define the mapping $m_b: S\times S\to[0,\infty)$ by

$$m_b(\zeta,\eta)=\begin{cases}0, & \zeta=\eta\in S\backslash\{1\}\\ 1, & (\zeta=0\text{and}\eta=1)\text{ or }(\zeta=1\text{ and }\eta=0)\\ \frac{1}{n}, & \begin{array}{l}\zeta=1\text{ and }\eta\in\{\frac{1}{n}:n\geq1,n\in\mathbb{N}\}\\ \text{or}\\ \zeta\in\{\frac{1}{n}:n\geq1,n\in\mathbb{N}\}\text{ and }\eta=1\end{array}\\ \max\{\zeta,\eta\}, & \zeta,\eta\in\{\frac{1}{n}:n>1,n\in\mathbb{N}\}\text{ and }\zeta\neq\eta\\ 2, & \text{otherwise}\end{cases}.$$

In this case, (S, m_b) is an M_b-metric space with the coefficient $s=2$. But it is not a strong M_b-metric space, since the property $sm4$) is not satisfied. Indeed, assume that the property $sm4$) is satisfied. Then, for $\zeta=0$, $\eta=\frac{1}{n}, n\geq2$ and $\xi=1$, we have

$$\begin{aligned}m_b(\zeta,\eta)-m_{b\zeta\eta}&=2\\ &\leq1+s.\frac{1}{n}\\ &=m_b(\zeta,\xi)-m_{b\zeta\xi}+s\big(m_b(\eta,\xi)-m_{b\eta\xi}\big).\end{aligned}$$

Thus, we get $2\leq1$ as $n\to\infty$. This is a contradiction. Therefore, (S, m_b) is not a strong M_b-metric space.

Let (S, m_b, s) be a strong M_b-metric space. A subset U of S is open iff there is a real number $\varepsilon>0$ such that the open ball $B(m_b,\zeta,\varepsilon)\subseteq U$ for all $\zeta\in U$ where $B(m_b,\zeta,\varepsilon)=\{\eta\in S: m_b(\zeta,\eta)<m_{b\zeta\eta}+\varepsilon\}$. If τ_{m_b} denotes the family of all open subsets of S, then it is a topology on S.

Definition 4 Let (S, m_b, s) be a strong M_b-metric space and $\{\zeta_n\}$ be a sequence in S. Then,

(1) The sequence $\{\zeta_n\}$ is said to M_b-converges to $\zeta\in S$ if and only if

$$\lim_{n\to\infty} m_b(\zeta_n,\zeta) - m_{b\zeta_n\zeta} = 0.$$

(2) $\{\zeta_n\}$ is called M_b-Cauchy sequence if

$$\lim_{n,m\to\infty} m_b(\zeta_n,\zeta_m)$$

exists and finite.

(3) (S, m_b) is said to be M_b-complete strong M_b-metric space if and only if every M_b-Cauchy sequence in this space converges to any point ζ in S and

$$\lim_{n,m\to\infty} m_b(\zeta_n,\zeta_m) = m_b(\zeta,\zeta).$$

Lemma 1 *Let (S, m_b, s) be a strong M_b-metric space and $\{\zeta_n\}$ be a sequence in S. Then, the sequence $\{\zeta_n\}$ converges to $\zeta \in S$ with respect to τ_{m_b} if and only if the sequence $\{\zeta_n\}$ M_b-converges to $\zeta \in S$.*

Proof Let $\zeta_n \xrightarrow{\tau_{m_b}} \zeta$ and $\varepsilon > 0$. In this case, there exists $n_0 \in \mathbb{N}$ such that $\zeta_n \in B(m_b, \zeta, \varepsilon)$ for all $n \geq n_0$. Therefore, we have $m_b(\zeta_n,\zeta) < m_{b\zeta_n\zeta} + \varepsilon$ and so $\left|m_b(\zeta_n,\zeta) - m_{b\zeta_n\zeta}\right| < \varepsilon$ for all $n \geq n_0$. Now, assume the opposite. Let $\lim_{n\to\infty} m_b(\zeta_n,\zeta) - m_{b\zeta_n\zeta} = 0$, $U \in \tau_{m_b}$ and $\zeta \in S$. From the definition of τ_{m_b}, there exists an open ball with centered $\zeta \in S$ and radius $\varepsilon > 0$ such that $B(m_b, \zeta, \varepsilon) \subseteq U$. Since $\lim_{n\to\infty} m_b(\zeta_n,\zeta) - m_{b\zeta_n\zeta} = 0$, for the $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $m_b(\zeta_n,\zeta) - m_{b\zeta_n\zeta} < \varepsilon$ for all $n \geq n_0$. Hence, we get $m_b(\zeta_n,\zeta) < m_{b\zeta_n\zeta} + \varepsilon$, that is, $\zeta_n \in B(m_b, \zeta, \varepsilon) \subseteq U$ for all $n \geq n_0$. Therefore, we get $\zeta_n \xrightarrow{\tau_{m_b}} \zeta$. □

Note that, every M_b-metric may not be continuous as in Example 2 . However, the following lemma shows that strong M_b-metric is a continuous mapping.

Lemma 2 *Let (S, m_b, s) be a strong M_b-metric space, $\{\zeta_n\}, \{\eta_n\}$ be sequences in S and $\zeta, \eta \in S$. If $\zeta_n \xrightarrow{\tau_{m_b}} \zeta$ and $\eta_n \xrightarrow{\tau_{m_b}} \eta$, then we have*

$$\lim_{n\to\infty} m_b(\zeta_n,\eta_n) - m_{b\zeta_n\eta_n} = m_b(\zeta,\eta) - m_{b\zeta\eta}.$$

Proof Let $\zeta_n \xrightarrow{\tau_{m_b}} \zeta$ and $\eta_n \xrightarrow{\tau_{m_b}} \eta$. Using the property *sm*4) twice in succession, we have

$$\begin{aligned} m_b(\zeta_n,\eta_n) - m_{b\zeta_n\eta_n} &\leq (m_b(\zeta_n,\eta) - m_{b\zeta_n\eta}) \\ &\quad + s\big(m_b(\eta_n,\eta) - m_{b\eta_n\eta}\big) \\ &\leq \big(m_b(\zeta,\eta) - m_{b\zeta\eta}\big) \\ &\quad + s\big[\big(m_b(\zeta_n,\zeta) - m_{b\zeta_n\zeta}\big) + \big(m_b(\eta_n,\eta) - m_{b\eta_n\eta}\big)\big] \end{aligned}$$

and so,

$$\left(m_b(\zeta_n,\eta_n)-m_{b\zeta_n\eta_n}\right)-\left(m_b(\zeta,\eta)-m_{b\zeta\eta}\right)\leq s\begin{bmatrix}\left(m_b(\zeta_n,\zeta)-m_{b\zeta_n\zeta}\right)\\+\left(m_b(\eta_n,\eta)-m_{b\eta_n\eta}\right)\end{bmatrix}$$

for all $n\in\mathbb{N}$. Similarly, it can be obtained that

$$\left(m_b(\zeta,\eta)-m_{b\zeta\eta}\right)-\left(m_b(\zeta_n,\eta_n)-m_{b\zeta_n\eta_n}\right)\leq s\begin{bmatrix}\left(m_b(\zeta_n,\zeta)-m_{b\zeta_n\zeta}\right)\\+\left(m_b(\eta_n,\eta)-m_{b\eta_n\eta}\right)\end{bmatrix}$$

for all $n\in\mathbb{N}$. Therefore, we get

$$\left|\left(m_b(\zeta_n,\eta_n)-m_{b\zeta_n\eta_n}\right)-\left(m_b(\zeta,\eta)-m_{b\zeta\eta}\right)\right|\leq s\begin{bmatrix}\left(m_b(\zeta_n,\zeta)-m_{b\zeta_n\zeta}\right)\\+\left(m_b(\eta_n,\eta)-m_{b\eta_n\eta}\right)\end{bmatrix}$$

for all $n\in\mathbb{N}$. Moreover, since $\zeta_n\overset{\tau_{m_b}}{\longrightarrow}\zeta$ and $\eta_n\overset{\tau_{m_b}}{\longrightarrow}\eta$, taking a limit as $n\to\infty$, we have

$$\lim_{n\to\infty}\left|\left(m_b(\zeta_n,\eta_n)-m_{b\zeta_n\eta_n}\right)-\left(m_b(\zeta,\eta)-m_{b\zeta\eta}\right)\right|=0$$

and thus,

$$\lim_{n\to\infty}\left(m_b(\zeta_n,\eta_n)-m_{b\zeta_n\eta_n}\right)=m_b(\zeta,\eta)-m_{b\zeta\eta}.$$

□

Remark 1 Note that, in Example 2, if we take $\zeta_n=\frac{1}{n}$, $\eta_n=0$ for all $n\in\mathbb{N}$, then although $\zeta_n\overset{\tau_{m_b}}{\longrightarrow}1$ and $\eta_n\overset{\tau_{m_b}}{\longrightarrow}0$, we get

$$\begin{aligned}\lim_{n\to\infty}\left(m_b(\zeta_n,\eta_n)-m_{b\zeta_n\eta_n}\right)&=\lim_{n\to\infty}\left(m_b\left(\frac{1}{n},0\right)-m_{b\frac{1}{n}0}\right)\\&=2\\&\neq 1\\&=m_b(1,0)-m_{b10}\\&=m_b(\zeta,\eta)-m_{b\zeta\eta}\end{aligned}$$

Hence, from Lemma (2), the mapping m_b is not a strong M_b -metric.

Before the main result, we give the following Lemma 3 and Lemma 4 , since they are very important in our main result.

Lemma 3 *Let (S,m_b,s) be a strong M_b-metric space. Then the function $\rho_{m_b}:S\times S\to[0,\infty)$ defined by*

$$\rho_{m_b}(\zeta,\eta)=m_b(\zeta,\eta)-2m_{b\zeta\eta}+M_{b\zeta\eta}$$

for all $\zeta,\eta\in S$ is a strong b-metric on S with the same coefficient s, where

$$M_{b\zeta\eta}=\max\{m_b(\zeta,\zeta),m_b(\eta,\eta)\}.$$

Proof Let (S, m_b, s) be a strong M_b-metric space. We need to show that all conditions of Definition of 1 are satisfied. The properties $m1$) and $m2$) are clear. So, it is enough to show the property $m3$). For $\zeta, \eta, \xi \in S$,

$$\begin{aligned}\rho_{m_b}(\zeta,\eta) &= m_b(\zeta,\eta) - 2m_{b\zeta\eta} + M_{b\zeta\eta} \\ &= \big(m_b(\zeta,\eta) - m_{b\zeta\eta}\big) + \big(M_{b\zeta\eta} - m_{b\zeta\eta}\big).\end{aligned}$$

Since, from $sm4$), we have

$$m_b(\zeta,\eta) - m_{b\zeta\eta} \le s(m_b(\zeta,\xi) - m_{b\zeta\xi}) + \big(m_b(\eta,\xi) - m_{b\eta\xi}\big)$$

and

$$\begin{aligned}M_{b\zeta\eta} - m_{b\zeta\eta} &= |m_b(\zeta,\zeta) - m_b(\eta,\eta)| \\ &\le |m_b(\zeta,\zeta) - m_b(\xi,\xi)| + |m_b(\xi,\xi) - m_b(\eta,\eta)| \\ &\le s|m_b(\zeta,\zeta) - m_b(\xi,\xi)| + |m_b(\xi,\xi) - m_b(\eta,\eta)| \\ &= s(M_{b\zeta\xi} - m_{b\zeta\xi}) + \big(M_{b\eta\xi} - m_{b\eta\xi}\big),\end{aligned}$$

it can be obtained

$$\begin{aligned}\rho_{m_b}(\zeta,\eta) &= \big(m_b(\zeta,\eta) - m_{b\zeta\eta}\big) + \big(M_{b\zeta\eta} - m_{b\zeta\eta}\big) \\ &\le s(m_b(\zeta,\xi) - m_{b\zeta\xi}) + \big(m_b(\eta,\xi) - m_{b\eta\xi}\big) \\ &\quad + s(M_{b\zeta\xi} - m_{b\zeta\xi}) + \big(M_{b\eta\xi} - m_{b\eta\xi}\big) \\ &= s\rho_{m_b}(\zeta,\xi) + \rho_{m_b}(\eta,\xi).\end{aligned}$$

□

Lemma 4 *Let (S, m_b, s) be a strong M_b-metric space and $\{\zeta_n\}$ be a sequence in S. In this case, we have*

(a) $\{\zeta_n\}$ is an M_b-Cauchy sequence iff $\{\zeta_n\}$ is a Cauchy sequence in the strong b-metric space (S, ρ_{m_b}, s).
(b) (S, m_b, s) is an M_b-complete strong M_b-metric space iff (S, ρ_{m_b}, s) is a complete strong b-metric space.

Proof of (a) Let $\{\zeta_n\}$ be an M_b-Cauchy sequence. Then, there exists $a \in \mathbb{R}$ such that

$$\lim_{n,k\to\infty} m_b(\zeta_n,\zeta_k) = a.$$

Since $\lim_{n,k\to\infty} m_b(\zeta_n,\zeta_k) = a$ implies $\lim_{n\to\infty} m_b(\zeta_n,\zeta_n) = a$, we have

$$\lim_{n,k\to\infty} \rho_{m_b}(\zeta_n,\zeta_k) = \lim_{n,k\to\infty} \big[m_b(\zeta_n,\zeta_k) - 2m_{b\zeta_n\zeta_k} + M_{b\zeta_n\zeta_k}\big] = 0.$$

Therefore, $\{\zeta_n\}$ is a Cauchy sequence in the strong b-metric space (S, ρ_{m_b}, s). Now, let $\{\zeta_n\}$ be a Cauchy sequence in (S, ρ_{m_b}, s). That is, we have

$$\lim_{n,k\to\infty} \rho_{m_b}(\zeta_n, \zeta_k) = 0.$$

Thus, for $\varepsilon = 1$, there exists $n_0 \in \mathbb{N}$ such that $\rho_{m_b}(\zeta_n, \zeta_k) < 1$ for all $n, k \geq n_0$. Therefore we have

$$\begin{aligned} m_b(\zeta_n, \zeta_n) &= m_b(\zeta_n, \zeta_n) - m_b(\zeta_{n_0}, \zeta_{n_0}) + m_b(\zeta_{n_0}, \zeta_{n_0}) \\ &\leq \left| m_b(\zeta_n, \zeta_n) - m_b(\zeta_{n_0}, \zeta_{n_0}) \right| + m_b(\zeta_{n_0}, \zeta_{n_0}) \\ &= M_{b\zeta_n\zeta_{n_0}} - m_{b\zeta_n\zeta_{n_0}} + m_b(\zeta_{n_0}, \zeta_{n_0}) \\ &\leq \rho_{m_b}(\zeta_n, \zeta_{n_0}) + m_b(\zeta_{n_0}, \zeta_{n_0}) \\ &< 1 + m_b(\zeta_{n_0}, \zeta_{n_0}). \end{aligned}$$

Thus, $\{m_b(\zeta_n, \zeta_n)\} \subseteq \mathbb{R}$ is a bounded sequence. Therefore, there exists a real number a such that the subsequence $\{m_b(\zeta_{n_k}, \zeta_{n_k})\}$ converges to a. On the other hand, since $\{\zeta_n\}$ is a Cauchy sequence in (S, ρ_{m_b}, s), for any $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$\begin{aligned} |m_b(\zeta_n, \zeta_n) - m_b(\zeta_k, \zeta_k)| &= M_{b\zeta_n\zeta_k} - m_{b\zeta_n\zeta_k} \\ &\leq \rho_{m_b}(\zeta_n, \zeta_k) \\ &< \varepsilon \end{aligned}$$

for all $n, k \geq n_1$. Thus $\{m_b(\zeta_n, \zeta_n)\}$ is a Cauchy sequence in $\mathbb{R}$ and so we get $m_b(\zeta_n, \zeta_n) \to a$, as $n \to \infty$. Now, since

$$\begin{aligned} |m_b(\zeta_n, \zeta_k) - a| &= \left| \begin{matrix} m_b(\zeta_n, \zeta_k) - 2m_{b\zeta_n\zeta_k} + M_{b\zeta_n\zeta_k} \\ \quad + 2m_{b\zeta_n\zeta_k} - M_{b\zeta_n\zeta_k} - a \end{matrix} \right| \\ &\leq \left| m_b(\zeta_n, \zeta_k) - 2m_{b\zeta_n\zeta_k} + M_{b\zeta_n\zeta_k} \right| \\ &\quad + \left| 2m_{b\zeta_n\zeta_k} - M_{b\zeta_n\zeta_k} - a \right| \\ &= \rho_{m_b}(\zeta_n, \zeta_k) + \left| 2m_{b\zeta_n\zeta_k} - M_{b\zeta_n\zeta_k} - a \right|, \end{aligned}$$

we get $\lim_{n,k\to\infty} m_b(\zeta_n, \zeta_k) = a$. Therefore, $\{\zeta_n\}$ is an M_b-Cauchy sequence. □

Proof of (b) Now, let (S, m_b, s) be an M_b-complete strong M_b-metric space and $\{\zeta_n\}$ be a Cauchy sequence in (S, ρ_{m_b}, s). Then, from the condition (a), the sequence $\{\zeta_n\}$ is an M_b-Cauchy sequence and so there exists $\zeta \in S$ such that

$$\lim_{n\to\infty} \left[m_b(\zeta_n, \zeta) - m_{b\zeta_n\zeta} \right] = 0$$

and

$$\lim_{n,m\to\infty} m_b(\zeta_n, \zeta_m) = m_b(\zeta, \zeta).$$

Therefore we have

$$\begin{aligned}\lim_{n\to\infty}\rho_{m_b}(\zeta_n,\zeta) &= \lim_{n\to\infty}\left[m_b(\zeta_n,\zeta)-2m_{b\zeta_n\zeta}+M_{b\zeta_n\zeta}\right]\\ &= \lim_{n\to\infty}\left[m_b(\zeta_n,\zeta)-m_{b\zeta_n\zeta}+M_{b\zeta_n\zeta}-m_{b\zeta_n\zeta}\right]\\ &= 0.\end{aligned}$$

Thus (S,ρ_{m_b},s) is a complete strong b-metric space. Now, we assume that (S,ρ_{m_b},s) is a complete strong b-metric space and $\{\zeta_n\}$ is an M_b-Cauchy sequence. Then, it is a Cauchy sequence in (S,ρ_{m_b},s) and hence, there exists $\zeta\in S$ such that

$$\lim_{n\to\infty}\rho_{m_b}(\zeta_n,\zeta)=\lim_{n\to\infty}\left[m_b(\zeta_n,\zeta)-m_{b\zeta_n\zeta}+M_{b\zeta_n\zeta}-m_{b\zeta_n\zeta}\right]=0.$$

Therefore since $m_b(\zeta_n,\zeta)-m_{b\zeta_n\zeta}\geq 0$ and $M_{b\zeta_n\zeta}-m_{b\zeta_n\zeta}\geq 0$, we get $\lim_{n\to\infty}\left[m_b(\zeta_n,\zeta)-m_{b\zeta_n\zeta}\right]=0$ and $\lim_{n\to\infty}\left[M_{b\zeta_n\zeta}-m_{b\zeta_n\zeta}\right]=0$. Hence, $\{\zeta_n\}$ M_b -converges to ζ. Since $\{\zeta_n\}$ be an M_b-Cauchy sequence in (S,m_b,s), it is sufficient to see that $\lim_{n\to\infty}m_b(\zeta_n,\zeta_n)=m_b(\zeta,\zeta)$. Since

$$\begin{aligned}0 &= \lim_{n\to\infty}\left[M_{b\zeta_n\zeta}-m_{b\zeta_n\zeta}\right]\\ &= \lim_{n\to\infty}|m_b(\zeta_n,\zeta_n)-m_b(\zeta,\zeta)|,\end{aligned}$$

we have $\lim_{n\to\infty}m_b(\zeta_n,\zeta_n)=m_b(\zeta,\zeta)$. □

The following example shows that Caristi fixed point result doesn't extend to strong M_b-metric spaces and so, to M_b-metric spaces neither.

Example 3 Let $S=\mathbb{N}$ be a set of natural numbers and define the mapping $m_b: S\times S\to[0,\infty)$ as in Example 1. Besides, let $T:S\to S$ and $\varphi:S\to[0,\infty)$ be mappings such that

$$T\zeta=\begin{cases}4, & \zeta=2\\ 2, & \zeta\in S\backslash\{2\}\end{cases}$$

and

$$\varphi\zeta=\begin{cases}2, & \zeta=2 \text{ or } \zeta=4\\ 3, & \zeta\in S\backslash\{2,4\}\end{cases}.$$

for all $\zeta\in S$. Then, we know from Example 1 that (S,m_b,s) is a strong M_b-metric space with the coefficient $s=2$.

(1) (S,m_b,s) is an M_b-complete strong M_b-metric space. *Indeed, let* $\{\zeta_n\}$ be an M_b-Cauchy sequence. In this case, there exists a real number α *such that* $\lim_{n,m\to\infty}m_b(\zeta_n,\zeta_m)=\alpha$ and so, we have $\lim_{n\to\infty}m_b(\zeta_n,\zeta_n)=\alpha$. Therefore, it has to be $\alpha=0$ *or* $\alpha=4$. However, there is no any sequence satisfying $\lim_{n,m\to\infty}m_b(\zeta_n,\zeta_m)=0$ *and* $\lim_{n,m\to\infty}m_b(\zeta_n,\zeta_m)=4$ except that the sequence whose all terms are fixed after a certain term. Thus, (S,m_b,s) is an M_b-complete strong M_b-metric space.

(2) It is clear that φ is lower semicontinuous with respect to τ_{m_b}. Actually, for all $\zeta \in S$ except that $\zeta = 2$ and $\zeta = 4$, the set $\{\zeta\}$ is open with respect to τ_{m_b}. Then, we have

$$\varphi(\zeta) \leq \sup_{r>0} \inf_{\eta \in B(m_b,\zeta,r)} \varphi(\eta) \tag{2.1}$$

for all $\zeta \in S\backslash\{2,4\}$. Besides, for $\zeta = 2$, $\zeta = 4$ and $r = \frac{1}{2}$, since $B(m_b, \zeta, r) = \{2,4\}$, it is satisfied Eq. (2.1).

(3) The inequality $m_b(\zeta, T\zeta) \leq \varphi(\zeta) - \varphi(T\zeta)$ is satisfied for all $\zeta \in S$. Indeed, we have three cases:

Case 1 Let $\zeta = 2$. Then, since $T\zeta = 4$, we have

$$\begin{aligned} m_b(\zeta, T\zeta) &= 0 \\ &= \varphi(2) - \varphi(4) \\ &= \varphi(\zeta) - \varphi(T\zeta) \end{aligned}$$

Case 2 Let $\zeta = 4$. Then, since $T\zeta = 2$, we have

$$\begin{aligned} m_b(\zeta, T\zeta) &= 0 \\ &= \varphi(4) - \varphi(2) \\ &= \varphi(\zeta) - \varphi(T\zeta) \end{aligned}$$

Case 3 Let $\zeta = n \in \mathbb{N}\backslash\{2,4\}$. Then, since $T\zeta = 2$, we have

$$\begin{aligned} m_b(\zeta, T\zeta) &= 1 \\ &= \varphi(n) - \varphi(2) \\ &= \varphi(\zeta) - \varphi(T\zeta). \end{aligned}$$

Thus, there exists a Caristi mapping T *on* M_b*-complete strong* M_b -metric space such that T doesn't have a fixed point.

As we noticed from Example 3 that every Caristi mapping on M_b -complete strong M_b-metric spaces may not have a fixed point. Therefore, we modified the definition of Caristi mapping on these spaces as follows:

Definition 5 Let (S, m_b, s) be a strong M_b-metric space and $\varphi : S \to [0, \infty)$ be a lower semicontinuous function with respect to $\tau_{\rho_{m_b}}$. If the mapping $T : S \to S$ satisfies

$$\max\{m_b(\zeta, T\zeta), M_{b\zeta T\zeta}\} \leq m_{b\zeta T\zeta} + \varphi(\zeta) - \varphi(T\zeta) \tag{2.2}$$

for all $\zeta \in S$, then T is called Caristi mapping of new type.

Theorem 2 *Let* (S, m_b, s) *be an* M_b*-complete strong* M_b*-metric space and* T *be a Caristi mapping of new type on* S*. Then,* T *has a fixed point.*

Proof Let (S, m_b, s) be an M_b-complete strong M_b-metric space and $T : S \to S$ be a Caristi mapping of new type. In this case, from Lemma 4, (S, ρ_{m_b}, s) is a complete strong b-metric space. Also, there exists a function $\varphi : S \to [0, \infty)$ which is lower semicontinuous with respect to $\tau_{\rho_{m_b}}$ and the mapping T satisfies inequality (2.2). Thus, we have

$$m_b(\zeta, T\zeta) \leq m_{b\zeta T\zeta} + \varphi(\zeta) - \varphi(T\zeta)$$

and

$$M_{b\zeta T\zeta} \leq m_{b\zeta T\zeta} + \varphi(\zeta) - \varphi(T\zeta)$$

for all $\zeta \in S$. Summing last inequalities, we obtain

$$M_{b\zeta T\zeta} + \ m_b(\zeta, T\zeta) \leq 2m_{b\zeta T\zeta} + 2\varphi(\zeta) - 2\varphi(T\zeta)$$

and so,

$$m_b(\zeta, T\zeta) - 2m_{b\zeta T\zeta} + M_{b\zeta T\zeta} \leq 2\varphi(\zeta) - 2\varphi(T\zeta).$$

for all $\zeta \in S$. If we define $\phi : S \to [0, \infty)$ by $\phi(\zeta) = 2\varphi(\zeta)$, then ϕ is a lower semicontinuous function with respect to $\tau_{\rho_{m_b}}$. Therefore, we have

$$\rho_{m_b}(\zeta, T\zeta) \leq \phi(\zeta) - \phi(T\zeta).$$

for all $\zeta \in S$. Thus, we can say that the mapping T has a fixed point by using Theorem 1. □

Note that, in Example 3, the mapping φ is lower semicontinuous with respect to $\tau_{\rho_{m_b}}$, since every single point set $\{\zeta\}$ is open with respect to $\tau_{\rho_{m_b}}$ for all $\zeta \in S$. However, the mapping T doesn't satisfy inequality (2.2). Indeed, if we take $\zeta = 2$, then $T\zeta = 4$ and so,

$$\begin{aligned} \max\{m_b(\zeta, T\zeta), M_{b\zeta T\zeta}\} &= \max\{m_b(2, 4), M_{b2,4}\} \\ &= 4 \\ &> 0 \\ &= m_{b\zeta T\zeta} + \varphi(\zeta) - \varphi(T\zeta). \end{aligned}$$

Although, all hypotheses of Theorem 2 except inequality (2.2) are satisfied, T doesn't have a fixed point. This situation shows how our definition of Caristi mapping of new type is important to prove Theorem 2.

Now, we give some Caristi type fixed point theorems by using Caristi mapping of new type on the strong M_b-metric space. Thus, we extend the results proved by Bae [6] and Suzuki [23].

Theorem 3 *Let (S, m_b, s) be an M_b-complete strong M_b-metric space, $\varphi : S \to [0, \infty)$ be a lower semicontinuous function with respect to $\tau_{\rho_{m_b}}$ and $\chi : S \to [0, \infty)$ be a function such that*

$$\sup\left\{\chi(\zeta) : \zeta \in S, \varphi(\zeta) \leq \inf_{w\in S}\varphi(w) + \lambda\right\} < \infty \tag{2.3}$$

for $\lambda > 0$. *If* $T : S \to S$ *satisfies*

$$\max\{m_b(\zeta, T\zeta), M_{b\zeta T\zeta}\} \leq m_{b\zeta T\zeta} + \chi(\zeta)(\varphi(\zeta) - \varphi(T\zeta)) \tag{2.4}$$

for all $\zeta \in S$, *then* T *has a fixed point in* S.

Proof If $\chi(\zeta) = 0$ for some $\zeta \in S$, then we obtain

$$\max\{m_b(\zeta, T\zeta), M_{b\zeta T\zeta}\} \leq m_{b\zeta T\zeta}.$$

Besides, using the properties of *sm*1) and *sm*4), we have $\zeta = T\zeta$. Now, let $\chi(\zeta) > 0$ for all $\zeta \in S$. Then, from 2.4, $\varphi(T\zeta) \leq \varphi(\zeta)$ for all $\zeta \in S$. Let define Z and μ for $\lambda > 0$ as follows:

$$Z = \left\{\zeta \in S : \varphi(\zeta) \leq \inf_{w\in S}\varphi(w) + \lambda\right\}$$

and

$$\mu = \sup_{w\in Z}\chi(w) < \infty$$

Since (S, m_b, s) is an M_b-complete strong M_b-metric space, from the Lemma 4, (S, ρ_{m_b}, s) is a complete strong b-metric space. Further, Z is a closed subset of S with respect to $\tau_{\rho_{m_b}}$, because of lower semicontinuity of φ. Thus, we have that (Z, ρ_{m_b}, s) is a complete strong b-metric space. Therefore, from the Lemma 4, we can say that (Z, m_b, s) is an M_b-complete strong M_b-metric space. Besides, since Z is a nonempty set and $\varphi(T\zeta) \leq \varphi(\zeta)$, it can be obtained $T(Z) \subseteq Z$. Also, we have

$$\max\{m_b(\zeta, T\zeta), M_{b\zeta T\zeta}\} \leq m_{b\zeta T\zeta} + \mu\varphi(\zeta) - \mu\varphi(T\zeta)$$

for all $\zeta \in Z$. If we define the mapping $\phi : Z \to [0, \infty)$ by $\phi(\zeta) = \mu\varphi(\zeta)$, then it is lower semicontinuous in (Z, ρ_{m_b}, s). Thus, since all hypotheses of Theorem 2 are satisfied, there exists a point $\xi \in Z \subseteq S$ such that $\xi = T\xi$. □

Following corollaries can be directly obtained by using Theorem 3.

Corollary 1 *Let* (S, m_b, s) *be an* M_b*-complete strong* M_b*-metric space,* $\varphi : S \to [0, \infty)$ *be a lower semicontinuous function w.r.t.* $\tau_{\rho_{m_b}}$ *and* $u : [0, \infty) \to [0, \infty)$ *be an upper semicontinuous function. If the mapping* $T : S \to S$ *satisfies*

$$\begin{aligned}\max\{m_b(\zeta, T\zeta), M_{b\zeta T\zeta}\} \leq\ & m_{b\zeta T\zeta} \\ & + \max\{u(\varphi(\zeta)), u(\varphi(T\zeta))\}(\varphi(\zeta) - \varphi(T\zeta))\end{aligned} \tag{2.5}$$

for all $\zeta \in S$, *then* T *has a fixed point in* S.

Corollary 2 *Let (S, m_b, s) be an M_b-complete strong M_b-metric space, $\varphi : S \to [0, \infty)$ be a lower semicontinuous function w.r.t. $\tau_{\rho_{m_b}}$ and $u : [0, \infty) \to [0, \infty)$ be a nondecreasing function. If the mapping $T : S \to S$ satisfies either*

$$\max\{m_b(\zeta, T\zeta), M_{b\zeta T\zeta}\} \leq m_{b\zeta T\zeta} + u(\varphi(\zeta)\{\varphi(\zeta) - \varphi(T\zeta)\} \tag{2.6}$$

or

$$\max\{m_b(\zeta, T\zeta), M_{b\zeta T\zeta}\} \leq m_{b\zeta T\zeta} + u(\varphi(T\zeta)\{\varphi(\zeta) - \varphi(T\zeta)\} \tag{2.7}$$

for all $\zeta \in S$, then T has a fixed point in S.

Corollary 3 *Let (S, m_b, s) be an M_b-complete strong M_b-metric space, $\varphi : S \to [0, \infty)$ be a lower semicontinuous function w.r.t. $\tau_{\rho_{m_b}}$ and $u : [0, \infty) \to [0, \infty)$ be an upper semicontinuous function. If the mapping $T : S \to S$ satisfies $m_b(\zeta, T\zeta) \leq \varphi(\zeta)$ and*

$$\max\{m_b(\zeta, T\zeta), M_{b\zeta T\zeta}\} \leq m_{b\zeta T\zeta} + u(m_b(\zeta, T\zeta))\{\varphi(\zeta) - \varphi(T\zeta)\} \tag{2.8}$$

for all $\zeta \in S$, then T has a fixed point in S.

3 Applications to ordinary metric spaces

As we said before, since every ordinary metric is a strong M_b-metric, we can obtain the following corollaries which include Caristi fixed point theorem and results of Suzuki, Bae et al. and Bae, respectively.

Corollary 4 [8] *Let (S, ρ) be a complete metric space and let $\varphi : S \to [0, \infty)$ be a lower semicontinuous function. If the mapping $T : S \to S$ satisfies inequality (2.2) for all $\zeta \in S$, then T has a fixed point in S.*

Corollary 5 [23] *Let (S, ρ) be a complete metric space, $\varphi : S \to [0, \infty)$ be a lower semicontinuous function and $\chi : S \to [0, \infty)$ be a function such that*

$$\sup\left\{\chi(\zeta) : \zeta \in S, \varphi(\zeta) \leq \inf_{w \in S} \varphi(w) + \lambda\right\} < \infty$$

for $\lambda > 0$. If $T : S \to S$ satisfies inequality (2.4) for all $\zeta \in S$, then T has a fixed point in S.

Corollary 6 [7] *Let (S, ρ) be a complete metric space, $\varphi : S \to [0, \infty)$ be a lower semicontinuous function and $u : [0, \infty) \to [0, \infty)$ be an upper semicontinuous function. If the mapping $T : S \to S$ satisfies inequality (2.5) for all $\zeta \in S$, then T has a fixed point in S.*

Corollary 7 [6] *Let (S, ρ) be a complete metric space, $\varphi : S \to [0, \infty)$ be a lower semicontinuous function and $u : [0, \infty) \to [0, \infty)$ be a nondecreasing function. If the mapping $T : S \to S$ satisfies either inequality (2.6) or inequality (2.7) for all $\zeta \in S$, then T has a fixed point in S.*

Corollary 8 [6] *Let (S, ρ) be a complete metric space, $\varphi : S \to [0, \infty)$ be a lower semicontinuous function and $u : [0, \infty) \to [0, \infty)$ be an upper semicontinuous function. If the mapping $T : S \to S$ satisfies $m_b(\zeta, T\zeta) \leq \varphi(\zeta)$ and inequality (2.8) for all $\zeta \in S$, then T has a fixed point in S.*

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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