Eur. Phys. J. Plus (2026) 141:678
https://doi.org/10.1140/epjp/s13360-026-07906-z

THE EUROPEAN PHYSICAL JOURNAL PLUS

Regular Article

Existence of global attractor for the velocity–vorticity model of the g-Navier–Stokes equations

Özge Kazar[1,a], **Meryem Kaya**[2,b]

[1] Department of Electronics and Automation, Vocational School of Technical Sciences, Hitit University, 19000 Çorum, Türkiye
[2] Faculty of Science, Department of Mathematics, Gazi University, 06500 Ankara, Türkiye

Received: 6 March 2026 / Accepted: 31 May 2026
© The Author(s) 2026

Abstract In this paper, we prove the existence of global attractors for the two-dimensional velocity–vorticity model of the g-Navier–Stokes equations. The proof is based on energy estimates in Sobolev spaces, which help us understand the long-term behavior of the system.

1 Introduction

In this article, we study the existence of global attractor for the velocity–vorticity model of the g-Navier–Stokes equations (gNSEs), which are particularly effective in modeling strongly rotating flows. In recent years, various velocity–vorticity models have been proposed and analyzed extensively in the context of fluid dynamics [1–7], with several numerical studies demonstrating their accuracy [3, 5, 8, 9]. The vorticity formulation has also been investigated for the Navier–Stokes–Voigt equations, as studied by Larios, Pei and Rebholz [1]. These models, originally introduced as smooth approximations of the classical NSEs, have become central to modern investigations in mathematical fluid mechanics [1, 10–12]. Building on these advancements, we focus on the velocity–vorticity model of the gNSEs, providing a proof for the existence and uniqueness of both weak and strong solutions under periodic boundary conditions [6]. In this paper, we extend our analysis by proving the existence of a global attractor for this system, which is essential for understanding the long-term behavior of the flow. The existence of a global attractor for dynamical systems has been a fundamental research problem in the field and has attracted the attention of mathematicians and physicists [12–17]. The system under consideration is defined in the domain $\Omega = (0,\ 1)^2 \subset R^2$ and is given by the following set of equations:

$$\frac{\partial v}{\partial t} - \nu\Delta_g v + \nu\left(\frac{\nabla g}{g}\cdot\nabla\right)v + w\mathbf{k}\times v + \nabla Q = f, \tag{1}$$

$$\frac{\partial w}{\partial t} - \nu\Delta_g w + \nu\left(\frac{\nabla g}{g}\cdot\nabla\right)w + (v\cdot\nabla)w = \nabla\times f + w\left(\frac{\nabla g}{g}\cdot v\right), \tag{2}$$

$$\nabla\cdot(gv) = 0, \qquad \nabla\cdot(gw) = 0, \tag{3}$$

$$v(x,0) = v_0(x), \qquad w(x,0) = w_0(x) \tag{4}$$

where $v = (v_1, v_2)$ is the velocity field, q is the pressure, $Q = \nabla\left(q + \frac{1}{2}|v|^2\right)$, $\nu > 0$ is viscosity constant, $w = v_{2x_1} - v_{1x_2}$ is a scalar vorticity. We define the g-Laplacian operator as follows (see [18]):

$$-\Delta_g v = -\frac{1}{g}(\nabla\cdot g\nabla)v = -\Delta v - \left(\frac{\nabla g}{g}\cdot\nabla\right)v.$$

Equation (2) is obtained by taking the curl of the gNSEs and is called the vorticity equation. Here, g satisfies the following requirements and is a sufficiently smooth function:

$$\begin{aligned} &g(x_1,x_2) \in C^{\infty}_{per}(\Omega),\\ &0 < n_0 \le g(x_1,x_2) \le N_0 \quad \text{for every} \quad (x_1,x_2)\in\Omega,\\ &\|\nabla g\|_{\infty} = \sup_{(x_1,x_2)\in\Omega} |\nabla g(x_1,x_2)| < \infty \end{aligned} \tag{5}$$

[a] e-mail: ozgekazar@gmail.com (corresponding author)
[b] e-mail: kayam06@gmail.com

Published online: 11 June 2026

here n_0 and N_0 are two constants. We will organize the article as follows. In Sect. 2, we introduce the necessary functional spaces and notations, following the framework presented by Roh in [18]. In Sect. 3, we present the proof of the existence of global attractor for the velocity–vorticity model.

2 Preliminaries and notations

In this section, we introduce the standard notations and function spaces that will be used throughout the paper. For a more detailed explanation of the function spaces and operators, see [18, 19]. In this paper, the constant c varies from line to line. $L^2(\Omega,\ g)$ denotes the Hilbert space with the inner product $(v,\ u)_g = \int_\Omega (v\cdot u)g dx$, the norm $\|v\|^2_{L^2(\Omega,\ g)} = (v,\ v)_g$. Similarly, the Hilbert spaces $H^1(\Omega,\ g)$ is defined with the scalar product $(v,\ u)_{H^1(\Omega,\ g)} = (v,\ u)_g + \sum_{i=1}^{2}\left(\frac{\partial v}{\partial x_i},\ \frac{\partial u}{\partial x_i}\right)_g$. It is known that the norms of the spaces $L^2(\Omega)$ and $L^2(\Omega,\ g)$ are equivalent due to the following inequalities:

$$\sqrt{n_0}\|v\|_{L^2(\Omega)} \le \|v\|_{L^2(\Omega,g)} \le \sqrt{N_0}\|v\|_{L^2(\Omega)} \tag{6}$$

where n_0 and N_0 positive constants. Function spaces are defined within the periodic setting for the gNSEs, as follows:

$$\begin{aligned}
V_1 &= \left\{v \in (C^\infty_{per}(\Omega))^2 : \nabla\cdot(gv) = 0,\ \int_\Omega v dx = 0\right\},\\
\mathbf{H}_g &= \text{the closure of } V_1 \text{ in } (L^2(\Omega,\ g))^2,\\
\mathbf{V}_g &= \text{the closure of } V_1 \text{ in } (H^1(\Omega,\ g))^2,\\
\mathbf{V}'_g &= \text{the dual space of } \mathbf{V}_g \text{ in 2D.}
\end{aligned}$$

Vorticity is mentioned as a scalar. The vorticity space is defined as:

$$\begin{aligned}
V_2 &= \left\{v \in C^\infty_{per}(\Omega) : \nabla\cdot(gv) = 0,\ \int_\Omega v\, dx = 0\right\},\\
H_g &= \text{the closure of } V_2 \text{ in } L^2(\Omega,\ g),\\
V_g &= \text{the closure of } V_2 \text{ in } H^1(\Omega,\ g),\\
V'_g &= \text{the dual space of } V_g,\\
H^{curl}_g &= \left\{f \in H_g : \nabla\times f \in L^2(\Omega,\ g)\right\}.
\end{aligned}$$

Similar to the Leray Projection $P_g : L^2_{per}(\Omega,\ g) \longrightarrow H_g$ define the orthogonal projection. The g-Stokes operator A_g has the following properties (see [18]).

Proposition 1 *([18]) The g-Stokes operator A_g holds the following properties : (i) A_g is a compact inverse, self adjoint, positive operator. (ii) There exist countable eigenvalues of A_g satisfying*

$$0 < \lambda_g \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots$$

where $\lambda_g = \frac{4\pi^2 n_0}{N_0}$ and λ_g is a lower bound for λ_1 and λ_1 is the smallest eigenvalue of A_g.

Theorem 2 *([18]) (g-Poincaré inequality) Assume that g satisfies (5). Then we have*

$$\frac{2\pi\sqrt{n_0}}{\sqrt{N_0}}\|v\|_{L^2} \le \|\nabla v\|_{L^2} \quad \text{for } v \in V_g$$

where $n_0 \le g(x) \le N_0$ for all $x \in \Omega$.

The bilinear operator $B_g : V_g \times V_g \to V'_g$ defines the following equality

$$B_g(v,u) = P_g(v\cdot\nabla)u. \tag{7}$$

We define the trilinear form b_g as follows:

$$b_g(v,u,w) = \sum_{i,j=1}^{2}\int_\Omega v_i(D_i u_j)w_j g dx = \left(P_g(v\cdot\nabla)u, w\right)_g \tag{8}$$

where $v,\ u,\ w \in V_g$. b_g has the following properties

$$b_g(v,u,w) = -b_g(v,w,u), \tag{9}$$

$$b_g(v,u,u) = 0 \tag{10}$$

for appropriate smooth functions $v \in H_g, u,\ w \in V_g$ [18, 20].

Lemma 1 *([18, 20]) The bilinear operator B_g defined in (7) satisfies the following inequalities;*

$$\left|\left\langle B_g(v,u), w\right\rangle_{V_g'}\right| \leq c\|v\|_{L^2}^{\frac{1}{2}}\|\nabla v\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}\|w\|_{L^2}^{\frac{1}{2}}\|\nabla w\|_{L^2}^{\frac{1}{2}} \tag{11}$$

for all $v,\ u,\ w \in V_g$.

Lemma 2 *([19]) (The Uniform Gronwall Lemma) Assume that $\frac{dk}{dt}$ is locally integrable on $]t_0, +\infty[$ and that m, n, and k are three positive locally integrable functions that satisfy conditions*

$$\frac{dk}{dt} \leq mk + n \quad for \quad t \geq t_0, \tag{12}$$

$$\int_t^{t+r} m(s)ds \leq b_1, \quad \int_t^{t+r} n(s)ds \leq b_2, \quad \int_t^{t+r} k(s)ds \leq b_3 \quad for \quad t \geq t_0, \tag{13}$$

where $r,\ b_1,\ b_2,\ b_3$ are positive constants. We get

$$k(t+r) \leq \left(\frac{b_3}{r} + b_2\right)\exp(b_1), \quad \forall t \geq t_0. \tag{14}$$

3 Existence of global attractor

Here, we begin using the Helmholtz–Leray projector P_g on the system (1)-(4) to produce the analogous functional differential equation shown below.

$$\frac{dv}{dt} + \nu A_g v + \nu C_g v + P_g(w\mathbf{k} \times v) = P_g f, \tag{15}$$

$$\frac{dw}{dt} + \nu A_g w + \nu C_g w + B_g(v,w) = P_g(\nabla \times f) + P_g\left(w(\frac{\nabla g}{g} \cdot v)\right), \tag{16}$$

$$v(0) = v_0, \qquad w(0) = w_0 \tag{17}$$

where $C_g v = P_g\left[\left(\frac{\nabla g}{g} \cdot \nabla\right)v\right]$ and $\langle C_g v,\ u\rangle = b_g\left(\frac{\nabla g}{g},\ v,\ u\right)$. Kazar and Kaya gave the proof of global well-posedness of the system (1)-(4) in [6]. Keep in mind that the constant c varies from line to line throughout the paper.

Theorem 3 *([6]) Suppose that $v_0 \in \boldsymbol{H}_g$ and $w_0 \in H_g$ and $f \in L^2(0,\ T; H_g^{curl})$. If $\nabla \cdot (gw) = 0$ is satisfied, the velocity–vorticity system has unique global weak solution $(v,\ w)$ in the sense of the definition of weak solution.*

Theorem 4 *([6]) For the initial data $v_0 \in \boldsymbol{V}_g$, $w_0 \in V_g$ and $f \in L^2(0,\ T; H_g^{curl})$, then there exists a unique strong solution $(v,\ w)$ in the sense of the definition of strong solution.*

The global well-posedness holds for the system (15)-(17). Next, we demonstrate that the system has continuous dependence. Let $S(t) : \mathbf{H}_g \times H_g \longrightarrow \mathbf{H}_g \times H_g$ be the solution semigroup generated by system (15)-(17).

Lemma 3 *Let $(v_{10},\ w_{10}),\ (v_{20},\ w_{20}) \in \boldsymbol{H}_g \times H_g$, then the following inequality holds*

$$\|S(t)(v_{10},\ w_{10}) - S(t)(v_{20},\ w_{20})\|_{\boldsymbol{H}_g \times H_g} \leq e^{-\beta t}\|(v_{10},\ w_{10}) - (v_{20},\ w_{20})\|_{\boldsymbol{H}_g \times H_g}$$

where $\beta = \min\{\beta_1,\ \beta_2\} > 0$ and β_1 and β_2 depend on the parameters $\nu,\ \lambda_g,\ n_0,\ \|\nabla g\|_\infty$.

Proof Let $(v_1(t),\ w_1(t)),\ (v_2(t),\ w_2(t))$ be two solutions of system (15)-(17) with two initial data $(v_{10},\ w_{10}),\ (v_{20},\ w_{20})$, respectively. Let $V(t) = v_1(t) - v_2(t),\ Y(t) = w_1(t) - w_2(t)$. Then $(V(t),\ Y(t))$ satisfy

$$\frac{dV}{dt} + \nu A_g V + \nu C_g V + P_g(w_1\mathbf{k} \times V) + P_g(Y\mathbf{k} \times v_2) = 0, \tag{18}$$

$$\frac{dY}{dt} + \nu A_g Y + \nu C_g Y + B_g(v_1, Y) - B_g(V, w_2) = P_g\left(Y\left(\frac{\nabla g}{g} \cdot v_1\right)\right) - P_g\left(w_2\left(\frac{\nabla g}{g} \cdot V\right)\right), \tag{19}$$

$$(V(0), Y(0)) = (v_{10} - v_{20}, w_{10} - w_{20}). \tag{20}$$

Taking the inner product of (18) with V and integrating over Ω, it follows that

$$\frac{d}{dt}\|V(t)\|_{L^2}^2 + 2\nu\|\nabla V(t)\|_{L^2}^2 \leq \left|2\nu b_g\left(\frac{\nabla g}{g}, V(t), V(t)\right)\right| + \left|2\langle P_g(Y\mathbf{k}(t) \times v_2(t)), V(t)\rangle_g\right|. \tag{21}$$

Next, using Cauchy–Schwarz, Young, Poincaré inequalities, and (11), we get

$$2\nu b_g\left(\frac{\nabla g}{g}, V(t), V(t)\right) \leq \frac{\nu}{4}\|\nabla V(t)\|_{L^2}^2 + \frac{4\nu\|\nabla g\|_\infty^2}{n_0^2}\|V(t)\|_{L^2}^2,$$
$$2\langle P_g(Y\mathbf{k}(t) \times v_2(t)), V(t)\rangle_g \leq \frac{\nu}{4}\|\nabla Y(t)\|_{L^2}^2 + \frac{1}{\nu}4c^2\frac{1}{\lambda_g}\|\nabla v_2(t)\|_{L^2}^2\|V(t)\|_{L^2}^2.$$

We obtain the following inequalities

$$\frac{d}{dt}\|V(t)\|_{L^2}^2 + 2\nu\|\nabla V(t)\|_{L^2}^2 \leq \frac{\nu}{4}\|\nabla Y(t)\|_{L^2}^2 + \frac{1}{\nu}4c^2\frac{1}{\lambda_g}\|\nabla v_2(t)\|_{L^2}^2\|V(t)\|_{L^2}^2 + \frac{\nu}{4}\|\nabla V(t)\|_{L^2}^2 + \frac{4\nu\|\nabla g\|_\infty^2}{n_0^2}\|V(t)\|_{L^2}^2. \tag{22}$$

In a similar manner, we multiply (19) by Y and integrate by over Ω, it follows that

$$\frac{d}{dt}\|Y(t)\|_{L^2}^2 + 2\nu\|\nabla Y(t)\|_{L^2}^2 \leq \left|2\nu b_g\left(\frac{\nabla g}{g}, Y(t), Y(t)\right)\right| + \left|2b_g(V(t), w_2(t), Y(t))\right| + \left|2\left\langle P_g\left(Y(t)\left(\frac{\nabla g}{g}\cdot v_1(t)\right)\right), Y(t)\right\rangle_g\right| + \left|2\left\langle P_g\left(w_2(t)\left(\frac{\nabla g}{g}\cdot V(t)\right)\right), Y(t)\right\rangle_g\right|. \tag{23}$$

Next, using Cauchy–Schwarz, Young, Poincaré inequalities, and (11), we write

$$\left|2\nu b_g\left(\frac{\nabla g}{g}, Y(t), Y(t)\right)\right| \leq \frac{\nu}{4}\|\nabla Y(t)\|_{L^2}^2 + \frac{4\nu\|\nabla g\|_\infty^2}{n_0^2}\|Y(t)\|_{L^2}^2,$$
$$|2b_g(V(t), w_2(t), Y(t))| \leq \frac{\nu}{4}\|\nabla V(t)\|_{L^2}^2 + \frac{1}{\nu}4c^2\frac{1}{\lambda_g}\|\nabla w_2(t)\|_{L^2}^2\|\nabla Y(t)\|_{L^2}^2,$$
$$\left|2\left\langle P_g\left(Y(t)\left(\frac{\nabla g}{g}\cdot v_1(t)\right)\right), Y(t)\right\rangle_g\right| \leq \frac{\nu}{4}\|\nabla Y(t)\|_{L^2}^2 + \frac{1}{\nu}4c^2\frac{\|\nabla g\|_\infty^2}{n_0^2}\|v_1(t)\|_{L^2}^2\|Y(t)\|_{L^2}^2$$

and

$$\left|2\left\langle P_g\left(w_2(t)\left(\frac{\nabla g}{g}\cdot V(t)\right)\right), Y(t)\right\rangle_g\right| \leq \frac{\nu}{4}\|\nabla Y(t)\|_{L^2}^2 + \frac{1}{\nu}4c^2\frac{\|\nabla g\|_\infty^2}{\lambda_g n_0^2}\|w_2(t)\|_{L^2}^2\|V(t)\|_{L^2}^2.$$

Using the above inequalities, we obtain the following inequalities

$$\frac{d}{dt}\|Y(t)\|_{L^2}^2+\frac{5\nu}{4}\|\nabla Y(t)\|_{L^2}^2\leq\frac{\nu}{4}\|\nabla V(t)\|_{L^2}^2+\frac{4\nu\|\nabla g\|_\infty^2}{n_0^2}\|Y(t)\|_{L^2}^2$$
$$+\frac{4c^2}{\nu\lambda_g}\|\nabla w_2(t)\|_{L^2}^2\|\nabla Y(t)\|_{L^2}^2$$
$$+\frac{4c^2\|\nabla g\|_\infty^2}{\nu n_0^2}\|v_1(t)\|_{L^2}^2\|Y(t)\|_{L^2}^2$$
$$+\frac{4c^2\|\nabla g\|_\infty^2}{\lambda_g\nu n_0^2}\|w_2(t)\|_{L^2}^2\|V(t)\|_{L^2}^2. \tag{24}$$

When the (22) and (24) inequalities are added side by side, we get

$$\begin{aligned}&\frac{d}{dt}\left(\|Y(t)\|_{L^2}^2+\|V(t)\|_{L^2}^2\right)\\&+\left(\lambda_g\nu-\frac{4c^2}{\nu}\|\nabla w_2(t)\|_{L^2}^2-\frac{4\nu\|\nabla g\|_\infty^2}{n_0^2}-\frac{4c^2\|\nabla g\|_\infty^2}{\nu n_0^2}\|v_1(t)\|_{L^2}^2\right)\|Y(t)\|_{L^2}^2\\&+\left(\frac{3\lambda_g\nu}{2}-\frac{4c^2}{\nu\lambda_g}\|\nabla v_2(t)\|_{L^2}^2-\frac{4\nu\|\nabla g\|_\infty^2}{n_0^2}-\frac{4c^2\|\nabla g\|_\infty^2}{\lambda_g\nu n_0^2}\|w_2(t)\|_{L^2}^2\right)\|V(t)\|_{L^2}^2\\&\leq 0.\end{aligned}$$

Then we have

$$\frac{d}{dt}\left(\|V(t)\|_{L^2}^2+\|Y(t)\|_{L^2}^2\right)+\beta\left(\|V(t)\|_{L^2}^2+\|Y(t)\|_{L^2}^2\right)\leq 0$$

where

$$\beta_1=\lambda_g\nu-\frac{4c^2}{\nu}\|\nabla w_2(t)\|_{L^2}^2-\frac{4\nu\|\nabla g\|_\infty^2}{n_0^2}-\frac{4c^2\|\nabla g\|_\infty^2}{\nu n_0^2}\|v_1(t)\|_{L^2}^2,$$
$$\beta_2=\frac{3\lambda_g\nu}{2}-\frac{4c^2}{\nu\lambda_g}\|\nabla v_2(t)\|_{L^2}^2-\frac{4\nu\|\nabla g\|_\infty^2}{n_0^2}-\frac{4c^2\|\nabla g\|_\infty^2}{\lambda_g\nu n_0^2}\|w_2(t)\|_{L^2}^2$$

and suppose that $\beta_1,\ \beta_2>0$ and

$$\beta=\min\{\beta_1,\ \beta_2\}.$$

We use the Gronwall inequality and arrive at the following result:

$$\|V(t)\|_{L^2}^2+\|Y(t)\|_{L^2}^2\leq e^{-\beta t}\left(\|V(0)\|_{L^2}^2+\|Y(0)\|_{L^2}^2\right).$$

This implies

$$\|S(t)(v_{10},\ w_{10})-S(t)(v_{20},\ w_{20})\|_{\mathbf{H}_g\times H_g}\leq e^{-\beta t}\|(v_{10},\ w_{10})-(v_{20},\ w_{20})\|_{\mathbf{H}_g\times H_g}.$$

Thus, the proof of the solution's continuous dependence on its initial value is given. □

We now recall some definitions and theorem required for the proof of the existence of a global attractor.

Definition 1 ([21]) We say that a set B is an absorbing set if for any bounded set X, there exists $t_0(x)$ such that

$$S(t)X\subset B,\qquad \text{for all } t\geq t_0(X).$$

If a semigroup possesses a compact absorbing set, we say that semigroup is dissipative.

Definition 2 ([21]) If B is an absorbing set as in Definition 1, then

$$w(B)=\underset{t\geq 0}{\cap}S(t)B.$$

Theorem 5 *([21]) If $S(t)$ is dissipative and B is compact absorbing set, then there exists a global attractor*

$$A = \omega(B).$$

Before proceeding with the proof of Theorem 6, we recall the following uniform bounds obtained in [6] for the weak solutions of (15)-(17):

$$K_1 = \sup_{s\in[0,T]} \|v(s)\|^2_{\mathbf{H}_g}, \qquad K_2 = \sup_{s\in[0,T]} \|w(s)\|^2_{H_g},$$

where K_1 and K_2 depend on v_0, f, ν, T and w_0, f, ν, T, respectively. Both constants are finite for any fixed $T > 0$ and will be used in the uniform Gronwall estimates below.

To establish the existence of a global attractor, we first show that the semigroup $S(t)$ is dissipative, namely that it possesses an absorbing set. We then prove that $S(t)$ is relatively compact [19, 21].

Theorem 6 *The system (15)-(17) possesses a global attractor A that is compact in $\boldsymbol{H}_g \times H_g$ and is maximal among all bounded invariant sets.*

Proof We will estimate the $\mathbf{H}_g$ norm and H_g norm of v and w, respectively. Since the proof of the boundedness for $\|w\|_{L^2}$ and $\|\nabla w\|_{L^2}$ needs to use the boundedness of $\|\nabla v\|_{L^2}$ and $\|v\|_{L^2}$. We start with estimating the $\|v\|_{L^2}$. Multiplying (15) by v and integrating the result equation by parts over Ω, we have

$$\frac{d}{dt}\|v(t)\|^2_{L^2} + 2\nu\|\nabla v(t)\|^2_{L^2} \le 2\nu\left|b_g\left(\frac{\nabla g}{g}, v(t), v(t)\right)\right| + \left|2\langle P_g f, v(t)\rangle_g\right|. \tag{25}$$

Using Cauchy–Schwarz, Young and Poincaré inequalities for the right-hand side of (25), we get

$$\frac{d}{dt}\|v(t)\|^2_{L^2} + \nu\|\nabla v(t)\|^2_{L^2} \le \frac{2\nu\|\nabla g\|^2_\infty}{n_0^2}\|v(t)\|^2_{L^2} + \frac{2}{\nu\lambda_g}\|P_g f\|^2. \tag{26}$$

For the second term in the left side for (26), the Poincaré inequalities are used. We write

$$\frac{d}{dt}\|v(t)\|^2_{L^2} + \left(\nu\lambda_g - \frac{2\nu\|\nabla g\|^2_\infty}{n_0^2}\right)\|v(t)\|^2_{L^2} \le \frac{2}{\nu\lambda_g}\|P_g f\|^2 \tag{27}$$

and

$$\frac{d}{dt}\|v\|^2_{L^2} + \alpha\|v\|^2_{L^2} \le \frac{2}{\nu\lambda_g}\|P_g f\|^2$$

where $\alpha = \nu\lambda_g - \frac{2\nu\|\nabla g\|^2_\infty}{n_0^2}$ and suppose that $\alpha > 0$. Applying the Gronwall's inequality, we get

$$\|v(t)\|^2_{L^2} \le \|v_0\|^2_{L^2}e^{-\alpha t} + \frac{2\|P_g f\|^2}{\alpha\nu\lambda_g}\left(1 - e^{-\alpha t}\right).$$

Thus, we have

$$\lim_{t\to\infty}\sup\|v(t)\|^2_{L^2} \le \rho_1^2, \quad \rho_1^2 = \frac{2\|P_g f\|^2}{\alpha\nu\lambda_g}. \tag{28}$$

There is an absorbing set B_0 in $\mathbf{H}_g$, which is any ball of $\mathbf{H}_g$ centered at 0 of radius $\rho_1^{'} > \rho_1$. If B is a bounded set of $\mathbf{H}_g$ that is contained in a ball of $\mathbf{H}_g$ with radius R and a center at 0, then $S(t)B \subset B(0, \rho_1^{'})$ for $t \ge t_1^*(B, \rho_1^{'})$, where

$$t_1^* = \frac{1}{\alpha}\ln\frac{R^2}{\rho_1^{'2} - \rho_1^2}. \tag{29}$$

We also draw conclusions from (26), after integration in t, we get

$$\|v(t+1)\|^2_{L^2} + \nu\int_t^{t+1}\|\nabla v(s)\|^2_{L^2}ds \le \frac{2\nu\|\nabla g\|^2_\infty}{n_0^2}\int_t^{t+1}\|v(s)\|^2_{L^2}ds$$

$$+ \|v(t)\|^2_{L^2} + \frac{2}{\nu\lambda_g}\|P_g f\|^2. \tag{30}$$

When the first term on the left side of (30) is neglected and consider with (28) and if $v_0 \in B_0 \subset B_{\mathbf{H}_g}(0,\ R)$ and $t \geq t_1^*(B,\ \rho_1^{'})$, we write

$$\int_t^{t+1} \|\nabla v(s)\|_{L^2}^2 ds \leq \left(\frac{4\|\nabla g\|_\infty^2}{n_0^2 \alpha \nu \lambda_g} + \frac{2}{\nu \lambda_g}\right) \|P_g f\|^2 + \frac{1}{\nu}\rho_1^{'2}. \tag{31}$$

Here K_3 is a positive constant defined as

$$K_3 = \left(\frac{4\|\nabla g\|_\infty^2}{n_0^2 \alpha \nu \lambda_g} + \frac{2}{\nu \lambda_g}\right) \|P_g f\|^2 + \frac{1}{\nu}\rho_1^{'2},$$

which depends only on the parameters ν, λ_g, n_0, $\|\nabla g\|_\infty$, $\|P_g f\|$, and ρ_1' (the radius of the absorbing set in $\mathbf{H}_g$), but not on time t.

Similarly, if taking inner product of the equation (16) with w and integrated over Ω, we obtain

$$\frac{d}{dt}\|w\|_{L^2}^2 + 2\nu\|\nabla w\|_{L^2}^2 \leq 2\nu \left| b_g\left(\frac{\nabla g}{g},\ w,\ w\right)\right| + 2\left|\langle \nabla \times P_g f,\ w\rangle_g\right| + \left|\left(P_g\left(w\left(\frac{\nabla g}{g}.v\right)\right),\ w\right)_g\right|.$$

Using Cauchy–Schwarz, Young and Poincaré inequalities, we see that

$$\frac{d}{dt}\|w\|_{L^2}^2 + \left(\frac{\nu\lambda_g}{2} - \frac{2\nu\|\nabla g\|_\infty^2}{n_0^2} - \frac{2c^2\|\nabla g\|_\infty^2}{\nu n_0^2}\frac{2\|P_g f\|^2}{\alpha\nu\lambda_g}\right)\|w\|_{L^2}^2 \leq \frac{2}{\nu}\|P_g f\|^2 \tag{32}$$

and

$$\frac{d}{dt}\|w\|_{L^2}^2 + \alpha_1\|w\|_{L^2}^2 \leq \frac{2}{\nu}\|P_g f\|^2 \tag{33}$$

where $\alpha_1 = \frac{\nu\lambda_g}{2} - \frac{2\nu\|\nabla g\|_\infty^2}{n_0^2} - \frac{2c^2\|\nabla g\|_\infty^2}{\nu n_0^2}\frac{2\|P_g f\|^2}{\alpha\nu\lambda_g}$ and suppose that $\alpha_1 > 0$. Neglecting the second term of the left-hand side of (32), applying the Gronwall's inequality, we get

$$\|w(t)\|_{L^2}^2 \leq \|w_0\|_{L^2}^2 e^{-\alpha_1 t} + \frac{2\|P_g f\|^2}{\alpha_1 \nu}\left(1 - e^{-\alpha_1 t}\right). \tag{34}$$

From (34), we write

$$\lim_{t\to\infty} \sup \|w(t)\|_{L^2}^2 \leq \rho_2^2, \quad \rho_2^2 = \frac{2\|P_g f\|^2}{\alpha_1 \nu}. \tag{35}$$

Thus, there is an absorbing set B_0 in H_g, which is any ball of H_g centered at 0 of radius $\rho_2^{'} > \rho_2$. If B in a bounded set of H_g, included in a ball $B(0, R)$ of H_g, centered at 0 of radius R, then $S(t)B \subset B(0,\ \rho_2^{'})$ for $t \geq t_2^*(B,\ \rho_2^{'})$

$$t_2^* = \frac{1}{\alpha_1} \ln \frac{R^2}{\rho_2^{'2} - \rho_2^2}. \tag{36}$$

Therefore, $S(t)$ has the absorbing set in $\mathbf{H}_g \times H_g$ for $\forall t_3^* \geq \max\left\{t_1^*,\ t_2^*\right\}$. Also, after integrating with respect to t in the (32) inequality, we get the following inequality

$$\|w(t+1)\|_{L^2}^2 + \frac{\nu}{2}\int_t^{t+1} \|\nabla w(s)\|_{L^2}^2 ds \leq \left(\frac{2\nu\|\nabla g\|_\infty^2}{n_0^2} + \frac{4c^2\|\nabla g\|_\infty^2}{\nu^2 n_0^2 \alpha \lambda_g}\|P_g f\|^2\right)\int_t^{t+1} \|w(s)\|_{L^2}^2 ds + \frac{2}{\nu}\|P_g f\|^2 + \|w(t)\|_{L^2}^2. \tag{37}$$

If we neglect the first term on the left-hand side of (37) and consider with (35) and if $v_0 \in B_0 \subset B_{H_g}(0,\ R)$ and $t \geq t_2^*(B,\ \rho_2^{'})$, we write

$$\int_t^{t+1} \|\nabla w(s)\|_{L^2}^2 ds \leq \left(\frac{8\|\nabla g\|_\infty^2}{n_0^2 \alpha_1 \nu} + \frac{8c^2\|\nabla g\|_\infty^2}{\nu^4 n_0^2 \alpha \alpha_1 \lambda_g} + \frac{4}{\nu^2} + \frac{2}{\alpha_1 \nu}\right)\|P_g f\|^2. \tag{38}$$

Here K_4 is a positive constant defined as

$$K_4 = \left(\frac{8\|\nabla g\|_\infty^2}{n_0^2 \alpha_1 \nu} + \frac{8c^2 \|\nabla g\|_\infty^2 \|P_g f\|^2}{\nu^4 n_0^2 \alpha_1 \lambda_g} + \frac{4}{\nu^2} + \frac{2}{\alpha_1 \nu} \right) \|P_g f\|^2,$$

which is time-independent and depends only on the problem data. Now we will give the estimate for the $\|\nabla v\|_{L^2}$. If we take the inner product of the equation (15) by $A_g v$ and integrate over Ω, we can obtain

$$\frac{d}{dt}\|\nabla v(t)\|_{L^2}^2 + 2\nu \left\| A_g v(t) \right\|_{L^2}^2 \leq 2\nu \left| b_g\left(\frac{\nabla g}{g},\, v(t),\, A_g v(t) \right) \right|$$

$$+ 2\left| \left(w\mathbf{k}(t) \times v(t),\, A_g v(t) \right)_g \right| + 2\left| \left\langle P_g f,\, A_g v(t) \right\rangle_g \right|.$$

The terms on the right-hand side of the above-mentioned inequality will be discussed as follows:

$$2\nu \left| b_g\left(\frac{\nabla g}{g},\, v(t),\, A_g v(t) \right) \right| \leq \frac{\nu}{2} \left\| A_g v(t) \right\|_{L^2}^2 + \frac{2\nu \|\nabla g\|_\infty^2}{n_0^2} \|\nabla v(t)\|_{L^2}^2,$$

$$2\left| \left\langle P_g f,\, A_g v(t) \right\rangle_g \right| \leq \frac{\nu}{2} \left\| A_g v(t) \right\|_{L^2}^2 + \frac{2}{\nu} \left\| P_g f \right\|^2,$$

$$2\left| \left(w\mathbf{k}(t) \times v(t),\, A_g v(t) \right)_g \right| \leq \frac{\nu}{2} \left\| A_g v(t) \right\|_{L^2}^2 + \frac{27c^4}{2\nu^3} \|w(t)\|_{L^2}^2 \|\nabla w(t)\|_{L^2}^2 \|\nabla v(t)\|_{L^2}^2$$

where we have used Cauchy–Schwarz, Young and (11) inequalities. Combining all the above estimates, we obtain the following inequality:

$$\frac{d}{dt}\|\nabla v(t)\|_{L^2}^2 + \frac{\nu}{2} \left\| A_g v(t) \right\|_{L^2}^2 \leq \left(\frac{2\nu \|\nabla g\|_\infty^2}{n_0^2} \right.$$

$$\left. + \frac{27c^4}{2\nu^3} \|w(t)\|_{L^2}^2 \|\nabla w(t)\|_{L^2}^2 \right) \|\nabla v(t)\|_{L^2}^2$$

$$+ \frac{2}{\nu} \left\| P_g f \right\|^2. \tag{39}$$

In [6], we proved that $w \in C\left(0,\, T; H_g\right) \cap L^2\left(0,\, T; V_g\right)$ because of w weak solution of the velocity–vorticity model of gNSEs. So we have

$$\underset{s \in [0,\, T]}{Sup} \|w(s)\|_{H_g}^2 \leq K_2$$

where K_2 depend on $w_0,\ f,\ \nu,\ T$. Using the above results and assuming that v_0 belongs to a bounded set B of $\mathbf{H}_g$ and that $t \geq t_1^*\left(B,\, \rho_1'\right)$, t_1^* as in (29), we apply the uniform Gronwall Lemma to (39) with $m,\ n,\ k$ replaced by

$$m = \frac{2\nu \|\nabla g\|_\infty^2}{n_0^2} + \frac{27c^4}{2\nu^3} \|w(t)\|_{L^2}^2 \|\nabla w(t)\|_{L^2}^2, \quad n = \frac{2}{\nu} \left\| P_g f \right\|^2, \quad k = \|\nabla v(t)\|_{L^2}^2.$$

Thanks to (35) and (38), we estimate the following inequality in the following way

$$\int_t^{t+1} m(s) ds = \int_t^{t+1} \left(\frac{2\nu \|\nabla g\|_\infty^2}{n_0^2} + \frac{27c^4}{2\nu^3} \|w(s)\|_{L^2}^2 \|\nabla w(s)\|_{L^2}^2 \right) ds \leq b_1,$$

$$\int_t^{t+1} n(s) ds = \int_t^{t+1} \frac{2}{\nu} \left\| P_g f \right\|^2 ds \leq b_2,$$

$$\int_t^{t+1} k(s) ds = \int_t^{t+1} \|\nabla v(s)\|_{L^2}^2 ds \leq b_3.$$

Thus, using the uniform Gronwall Lemma, we get $\forall t \geq t_3^* + 1$

$$\|\nabla v(t)\|_{L^2}^2 \leq \left(\int_t^{t+1} \|\nabla v(s)\|_{L^2}^2 ds \right.$$

$$+\int_t^{t+1}\frac{2}{\nu}\left\|P_g f\right\|^2 ds\Bigg)\exp\left(\frac{2\nu\|\nabla g\|_\infty^2}{n_0^2}+\frac{27c^4}{2\nu^3}K_2K_4\right)$$

$$\leq\left(K_3+\frac{2}{\nu}\left\|P_g f\right\|^2\right)\exp\left(\frac{2\nu\|\nabla g\|_\infty^2}{n_0^2}+\frac{27c^4}{2\nu^3}K_2K_4\right). \tag{40}$$

Now we will give the estimate for the $\|\nabla w\|_{L^2}$. If we take the inner product of the equation (16) by $A_g w$ and integrated over Ω, we can write

$$\frac{d}{dt}\|\nabla w(t)\|_{L^2}^2+2\nu\left\|A_g w(t)\right\|_{L^2}^2\leq 2\nu\left|b_g\left(\frac{\nabla g}{g},w(t),A_g w(t)\right)\right|$$

$$+2\left|b_g\big(v(t),w(t),A_g w(t)\big)\right|$$

$$+2\left|\left(P_g\left(w(t)\left(\frac{\nabla g}{g}\cdot v(t)\right)\right),A_g w(t)\right)_g\right|$$

$$+2\left|\big\langle\nabla\times P_g f,A_g w(t)\big\rangle_g\right|. \tag{41}$$

We will consider the terms on the right-hand side of the inequality as follows:

$$2\nu\left|b_g\left(\frac{\nabla g}{g},\,w(t),\,A_g w(t)\right)\right|\leq\frac{\nu}{4}\left\|A_g w(t)\right\|_{L^2}^2+\frac{4\nu\|\nabla g\|_\infty^2}{n_0^2}\|\nabla w(t)\|_{L^2}^2,$$

$$2\left|b_g\big(v(t),\,w(t),\,A_g w(t)\big)\right|\leq\frac{\nu}{2}\left\|A_g w\right\|_{L^2}^2+\frac{27c^4}{2\nu^3}\|v(t)\|_{L^2}^2\|\nabla v(t)\|_{L^2}^2\|\nabla w(t)\|_{L^2}^2,$$

$$2\left|\left(P_g\left(w(t)\left(\frac{\nabla g}{g}\cdot v(t)\right)\right),\,A_g w(t)\right)_g\right|\leq\frac{\nu}{4}\left\|A_g w(t)\right\|_{L^2}^2$$

$$+\frac{4c^2\|\nabla g\|_\infty^2}{\lambda_g\nu n_0^2}\|\nabla v(t)\|_{L^2}^2\|\nabla w(t)\|_{L^2}^2$$

and

$$2\left|\big\langle\nabla\times P_g f,\,A_g w(t)\big\rangle_g\right|\leq\frac{\nu}{4}\left\|A_g w(t)\right\|_{L^2}^2+\frac{4}{\nu}\left\|\nabla\times P_g f\right\|^2$$

where we have used Hölder's, Young, Poincaré inequality and (11). If we consider the inequalities we obtained above, instead of the terms on the right side of (41), we obtain

$$\frac{d}{dt}\|\nabla w(t)\|_{L^2}^2+\frac{3\nu}{4}\left\|A_g w(t)\right\|_{L^2}^2\leq\left(\frac{4\nu\|\nabla g\|_\infty^2}{n_0^2}+\frac{27c^4}{2\nu^3}\|v(t)\|_{L^2}^2\|\nabla v(t)\|_{L^2}^2\right.$$

$$\left.+\frac{4c^2\|\nabla g\|_\infty^2}{\lambda_g\nu n_0^2}\|\nabla v(t)\|_{L^2}^2\right)\|\nabla w(t)\|_{L^2}^2+\frac{4}{\nu}\left\|\nabla\times P_g f\right\|^2. \tag{42}$$

In [6], we proved that $v\in C\big(0,\,T;\mathbf{H}_g\big)\cap L^2\big(0,\,T;\mathbf{V}_g\big)$ because of v weak solution of the velocity–vorticity model of gNSEs. So we have

$$\underset{s\in[0,\,T]}{Sup}\ \|v(s)\|_{\mathbf{H}_g}^2\leq K_1$$

where K_1 depend on $v_0,\ f,\ \nu,\ T$. Using the above results and assuming that w_0 belongs to a bounded set B of H_g and that $t\geq t_2^*\big(B,\,\rho_2'\big)$, t_2^* as in (36), we apply the uniform Gronwall Lemma to (42) with $m,\ n,\ k$ replaced by

$$m=\frac{4\nu\|\nabla g\|_\infty^2}{n_0^2}+\frac{27c^4}{2\nu^3}\|v(t)\|_{L^2}^2\|\nabla v(t)\|_{L^2}^2+\frac{4c^2\|\nabla g\|_\infty^2}{\lambda_g\nu n_0^2}\|\nabla v(t)\|_{L^2}^2,$$

$$n=\frac{4}{\nu}\left\|\nabla\times P_g f\right\|^2,\quad k=\|\nabla w(t)\|_{L^2}^2.$$

Thanks to (28) and (31) we estimate the following inequality in the following way

$$\int_t^{t+1} m(s)ds = \int_t^{t+1} \left(\frac{4\nu\|\nabla g\|_\infty^2}{n_0^2} + \frac{27c^4}{2\nu^3}\|v(s)\|_{L^2}^2\|\nabla v(s)\|_{L^2}^2 + \frac{4c^2\|\nabla g\|_\infty^2}{\lambda_g \nu n_0^2}\|\nabla v(s)\|_{L^2}^2 \right) ds$$

$$\leq \frac{4\nu\|\nabla g\|_\infty^2}{n_0^2} + \frac{27c^4}{2\nu^3}K_1K_3 + \frac{4c^2\|\nabla g\|_\infty^2}{\lambda_g \nu n_0^2}K_3 \leq b_1,$$

$$\int_t^{t+1} n(s)ds = \int_t^{t+1} \frac{4}{\nu}\left\|\nabla \times P_g f\right\|^2 ds \leq b_2,$$

$$\int_t^{t+1} k(s)ds = \int_t^{t+1} \|\nabla w(s)\|_{L^2}^2 ds \leq b_3.$$

Next, for all $t \geq t_3^* + 1$, we write

$$\|\nabla w(t)\|_{L^2}^2 \leq \left(\int_t^{t+1} \|\nabla w(s)\|_{L^2}^2 ds + \frac{4}{\nu}\left\|\nabla \times P_g f\right\|^2 \right) \exp\left(\frac{4\nu\|\nabla g\|_\infty^2}{n_0^2} + \frac{27c^4}{2\nu^3}K_1K_3 + \frac{4c^2\|\nabla g\|_\infty^2}{\lambda_g \nu n_0^2}K_3 \right)$$

$$\leq \left(K_4 + \frac{4}{\nu}\left\|\nabla \times P_g f\right\|^2 \right) \exp\left(\frac{4\nu\|\nabla g\|_\infty^2}{n_0^2} + \frac{27c^4}{2\nu^3}K_1K_3 + \frac{4c^2\|\nabla g\|_\infty^2}{\lambda_g \nu n_0^2}K_3 \right). \tag{43}$$

Denote by ρ_3^2 the right-hand side of (40). We conclude that the ball $B_{\mathbf{V}_g}(0,\ \rho_3)$ of $\mathbf{V}_g$, denoted by B_1, is an absorbing set in $\mathbf{V}_g$ for the semigroup $S(t)$. Furthermore, if B is any bounded set of $\mathbf{H}_g$, then $S(t)B \subset B_1$ for $t_3 \geq t_1^*(B,\ \rho_1') + 1$. Similarly denote by ρ_4^2 the right-hand side of (43). We conclude that the ball $B_{V_g}(0,\ \rho_4)$ of V_g, denoted by B_2, is an absorbing set in V_g for the semigroup $S(t)$. Furthermore, if B is any bounded set of H_g, then $S(t)B \subset B_2$ for $t_3 \geq t_2^*(B,\ \rho_2') + 1$. By the compactness of the embedding $\mathbf{V}_g \times V_g \hookrightarrow \mathbf{H}_g \times H_g$, we find that $\{S(t)\}_{t\geq 0}$ has a $\mathbf{H}_g \times H_g$ – global attractor A. □

4 Conclusion

In this study, the existence of a global attractor for the velocity–vorticity model of the g-Navier–Stokes equations in the two-dimensional case was proven. The analysis was based on energy estimates in Sobolev spaces to establish both the boundedness and the compactness of the semigroup generated by the system. The results demonstrate the dissipation of the system and the presence of a global attractor, which is compact and maximal among all bounded invariant sets. This work contributes to the mathematical understanding of the g-Navier–Stokes equations, particularly their velocity–vorticity formulations, and provides a foundation for further exploration of their dynamics under different physical and numerical conditions. Future research may extend these results to explore applications in fluid dynamics and related computational models.

Funding Open access funding provided by the Scientific and Technological Research Council of Türkiye (TÜBİTAK).

Data availability No data were used to support the findings of this study.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. A. Larios, Y. Pei, L. Rebholz, Global well-posedness of the velocity-vorticity-Voigt model of the 3D Navier-Stokes equations. J. Diff. Equ. **266**, 2435–2465 (2019)
2. H.K. Lee, M.A. Olshanskii, L.G. Rebholz, On error analysis for the 3D Navier-Stokes equations in velocity-vorticity-helicity form. SIAM J. Numer. Anal. **49**, 711–732 (2011)
3. M. Akbas, L.G. Rebholz, C. Zerfas, Optimal vorticity accuracy in an efficient velocity-vorticity method for the 2D Navier-Stokes equations. Calcolo **55**, 1–29 (2018)
4. M. Gardner, A. Larios, L.G. Rebholz, D. Vargun, C. Zerfas, Continuous data assimilation applied to a velocity-vorticity formulation of the 2D Navier-Stokes equations. Electron. Res. Arch. **29**, 2223–2247 (2020)
5. M.A. Olshanskii, L.G. Rebholz, Velocity-vorticity-helicity formulation and a solver for the Navier-Stokes equations. J. Comput. Phys. **229**, 4291–4303 (2010)
6. Ö. Kazar, M. Kaya, On the weak and strong solutions of the velocity-vorticity model of the g-Navier-Stokes equations. Turk. J. Math. **47**, 1694–1714 (2023)
7. T.B. Gatski, Review of incompressible fluid flow computations using the vorticity-velocity formulation. Appl. Numer. Math. **7**, 227–239 (1991)
8. A.J. Majda, A.L. Bertozzi, *Vorticity and Incompressible Flow, Cambridge Texts in Applied Mathematics*, vol. 27 (Cambridge University Press, Cambridge, 2002)
9. T. Heister, M.A. Olshanskii, L.G. Rebholz, Unconditional long-time stability of a velocity-vorticity method for the 2D Navier-Stokes equations. Numer. Math. **135**, 143–167 (2017)
10. E.S. Baranovskii, Strong solutions of the incompressible Navier-Stokes-Voigt model. Mathematics **8**, 181 (2020)
11. E.S. Baranovskii, The Navier-Stokes-Voigt equations with position-dependent slip boundary conditions. Z. Angew. Math. Phys. **74**, 6 (2023)
12. G. Yue, J. Wang, Attractors of the velocity-vorticity-Voigt model of the 3D Navier-Stokes equations with damping. Comput. Math. Appl. **80**, 434–452 (2020)
13. G. Raugel, G.R. Sell, Navier-Stokes equations on thin 3D domains. I. Global attractors and global regularity of solutions. J. Am. Math. Soc. **6**, 503–568 (1993)
14. J. Jiang, Y. Hou, The global attractor of g-Navier-Stokes equations with linear dampness on $\mathbb{R}^2$. Appl. Math. Comput. **215**, 1068–1076 (2009)
15. V.K. Kalantarov, E.S. Titi, Global attractors and determining modes for the 3D Navier-Stokes-Voigt equations. Chin. Ann. Math. Ser. B **30**, 697–714 (2009)
16. W. Layton, Existence of smooth attractors for the Navier-Stokes-omega model of turbulence. J. Math. Anal. Appl. **366**, 81–89 (2010)
17. X.L. Song, J.H. Wu, Existence of global attractors for two-dimensional Newton-Boussinesq equation. Nonlinear Anal. **157**, 1–19 (2017)
18. J. Roh, g-Navier-Stokes Equations, PhD Thesis, University of Minnesota, Minnesota (2001)
19. R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, vol. 68 (Springer-Verlag, New York, 1997)
20. R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis* (North-Holland, Amsterdam, 1984)
21. J. Robinson, Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors, Vol. 28 Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge (2001)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.