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On multiplicative order convergence and multiplicative order continuous operators

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Abstract: Let A and B be two f-algebras. This paper establishes the theoretical framework for multiplicative order convergence, clarifying its definition, properties, and relations to other convergence types. This includes a detailed discussion on the conditions under which multiplicative order convergence and order convergence, as well as unbounded order convergence, mutually imply each other. We establish an extension theorem for multiplicative order continuous operators, similar to Veksler's theorem for order continuous operators. As a result of this theorem, we derive the order structure of the space of multiplicative order continuous operators. We show that the space $L_{mo}(A,B)$ of order bounded unbounded multiplicative order continuous operators is an ideal in $L_b(A,B)$ for semiprime f-algebra A and Dedekind complete f-algebra B. We also provide an example showing that $L_{mo}(A,B)$ is not generally a band in $L_b(A,B)$, and identify conditions on A or B under which it becomes one. Finally, we establish the relationships between multiplicative order continuous operators and order continuous operators.

Key words: Riesz space, f-algebra, order convergent net, unbounded convergent net, multiplicative order convergent net

1. Introduction

Convergence and continuity are fundamental concepts in the study of mathematical structures, especially within functional analysis. In this context, Riesz spaces and Riesz algebras provide a unified framework for analyzing ordered vector spaces with both algebraic and topological structures. These structures deepen our understanding of abstract algebraic theories and have applications in areas such as economics, optimization, and differential equations.

We refer to the standard textbooks for the fundamental concepts of Riesz spaces [2, 14, 18, 21]. Throughout this paper, all Riesz spaces are assumed to be Archimedean. First, we recall some definitions and relevant information used in this work.

Definition 1.1 *A net $(\varkappa_\alpha)$ in a Riesz space R is called:*

- *order convergent to $\varkappa \in R$, if there is a net $(r_\beta)_{\beta\in B}$ in R with $r_\beta \downarrow 0$, and for every β, there exists α_0 such that $|\varkappa - \varkappa_\alpha| \leq r_\beta$ for all $\alpha \geq \alpha_0$ (denotation $\varkappa_\alpha \overset{o}{\to} \varkappa$);*
- *unbounded order convergent to $\varkappa \in R$, if $|\varkappa_\alpha - \varkappa| \wedge r \xrightarrow{o} 0$ for all $r \in R_+$ (denotation $\varkappa_\alpha \overset{uo}{\to} \varkappa$).*

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Several notions of order convergence have been introduced in the literature, and their interrelationships are discussed in [1]. The concept of unbounded order convergence serves as a generalization of the idea of almost everywhere convergence within function spaces. It can be traced back to the foundational work in [16] by Nakano, marking an important step in the development of this mathematical idea. Later, the term "unbounded order convergence" was formally introduced by R. De Marr in [15], solidifying its place in the literature. A key property of unbounded order convergence is that, for a net $(\varkappa_\alpha)$, it corresponds to coordinate-wise convergence in the spaces c_0 and l_p $(1 \leq p \leq \infty)$. Furthermore, in the $L_p(\mu)$ space $(1 \leq p < \infty)$, the uo-convergence of a sequence is equivalent to almost everywhere convergence. These relationships significantly broaden the scope of potential applications, making the study of unbounded order convergence increasingly relevant. As a result, this concept has been rigorously investigated in recent years, with numerous studies contributing to its development and applications [9, 10, 19, 20].

Remark 1.2 *The following statements are true.*

(1) If R is a Dedekind complete Riesz space and $(\varkappa_\alpha) \subseteq R$ is a net, then instead of (r_β) net in order convergence definition, we can choose (r_α) which has the same index as $(\varkappa_\alpha)$ [1].

(2) $\varkappa_\alpha \xrightarrow{o} 0$ in R implies $\varkappa_\alpha \xrightarrow{uo} 0$, the converse is not true. Let $R = c_0$, the Riesz space of all real-valued null sequences, and (e_n) be the standard basis of R. Then (e_n) uo-converges to zero, but it is not order convergent to zero.

(3) If a net is order bounded, then o-convergence and uo-convergence coincide.

(4) Order convergence and unbounded order convergence are equivalent in a Riesz space R iff R is a finite dimension space [9, 19].

A Riesz space A under an associative multiplication is said to be a Riesz algebra whenever the multiplication makes A an algebra (with the usual properties), and in addition, it satisfies the following property: $a, b \in A_+$ implies $a \cdot b \in A_+$. A Riesz algebra A is called f-algebra if A has the additional property that $a \wedge b = 0$ implies $(a \cdot c) \wedge b = (c \cdot a) \wedge b = 0$ for all $c \in A_+$. Every Archimedean f-algebra is a commutative algebra. Algebra A is called semiprime if the only nilpotent element in A is the null element. Clearly, A is semiprime iff $a^2 = 0$ in A implies $a = 0$. We present some fundamental properties of f-algebras that are essential for our paper, whose proofs can be found in [8, 11, 12, 17].

Remark 1.3 *Let A be an Archimedean f-algebra. Then, the following statements are true.*

(1) $a \perp b$ (i.e., $|a| \wedge |b| = 0$) implies $c \cdot a \perp b$, $a \cdot c \perp b$ for all $c \in A$;

(2) $c \cdot (a \vee b) = c \cdot a \vee c \cdot b$, $(a \vee b) \cdot c = a \cdot c \vee b \cdot c$ for all $c \in A_+$ and similarly for the infimum;

(3) $|a \cdot b| = |a| \cdot |b|$ for all $a, b \in A$;

(4) $a \perp b$ implies $a \cdot b = 0$;

(5) $0 \leq a \leq b \Rightarrow a^2 \leq b^2$;

(6) If A is semiprime $a, b \in A^+$, then $a^2 \leq b^2 \Rightarrow a \leq b$;

(7) If A is semiprime, then $a \perp b$ if and only if $a \cdot b = 0$;

(8) If $b_\alpha \xrightarrow{o} b$ in A, then $b_\alpha \cdot a = a \cdot b_\alpha \xrightarrow{o} a \cdot b = b \cdot a$ for all $a \in A$.

Aydın initially introduced the definition of multiplicative order convergence in [3]. Then, the concept of multiplicative order convergence was studied in normed Riesz algebras and Riesz algebras [4, 6, 7].

Definition 1.4 *Let A be a commutative Riesz algebra. A net (b_α) in A is said to be multiplicative order convergent to $b \in A$ (in short mo-convergent) if $|b_\alpha - b| \cdot a \xrightarrow{o} 0$ for all $a \in A_+$, and it is denoted by $b_\alpha \xrightarrow{mo} b$.*

From the definition, the following properties are easily obtained in the structure of f-algebras.

- o-convergence implies mo-convergence;
- It is also known that o-convergence is equivalent to mo-convergence in f-algebras with a multiplicative unit.

In [5], mo-convergence was studied as a special case within a broader framework that unifies and extends the techniques and results developed in the recent literature on unbounded convergences. By Proposition 2.3 and Theorem 8 of [5], mo-convergence is a linear full lattice convergence. Therefore, in appropriate cases, we can take the net (b_α) to be positive and converge to zero without loss of generality.

Let R and F be Riesz spaces. We denote the set of all order bounded operators from R into F by $L_b(R,F)$. Let $\Psi : R \to F$ between two Riesz spaces is said to be order continuous (in short o-continuous) whenever $\varkappa_\alpha \xrightarrow{o} 0$ in R implies $\Psi\varkappa_\alpha \xrightarrow{o} 0$ in F. The space of order continuous operators is denoted by $L_n(R,F)$. The properties of the space $L_n(R,F)$ are well known [2, 18, 21]. The notion of unbounded order continuous (in short uo-continuous) operator spaces is defined analogously. The space of order bounded uo-continuous operators is denoted by $L_{uo}(R,F)$. The relationships among these types of convergence and continuity have been extensively studied in the literature [19, 20].

2. Multiplicative order convergence

The association between o-convergence and uo-convergence is given in Remark 1.2. In this section, we establish the relationships between these two types of convergence and mo-convergence. We begin by examining the relationship between o-convergence and mo-convergence. As previously noted, o-convergence implies mo-convergence; however, the converse does not hold in general.

Example 2.1 *Let $A = \mathbb{R}^2$ with the coordinatewise ordering and as multiplication defined by*

$$(x_1, x_2) * (y_1, y_2) = (x_1 y_1, 0).$$

It is known that A is an Archimedean, commutative, nonsemiprime Riesz algebra [8, Example 1.6]. Let $(x_n) = (\frac{1}{n}, n)$ be a sequence in A. (x_n) is mo-convergent to zero, but it is not o-convergent to zero.

Example 2.2 *Let $A = c_0$ with the coordinatewise ordering and the coordinatewise multiplication. A is an Archimedean, commutative, semiprime f-algebra without a unit element. Let (e_n) be the standard basis of A. Then (e_n) is mo-convergent to zero, but it is not o-convergent to zero. It is known that (e_n) is not o-convergent to zero in c_0. Now we will show that (e_n) is mo-convergent to zero. Let $0 \leq (a_m) = (a_1, a_2, ..., a_m, ...)$ be a sequence in c_0 and let $b_1 = (a_1, a_2, ..., a_m, ...)$, $b_2 = (0, a_2, ..., a_m, ...), ...$, $b_n = (0, 0, ..., a_n, ...)$. It is clear that the sequence (b_n) in c_0 and $(b_n) \downarrow 0$. Also, since $(e_n) \cdot a \leq b_n \downarrow 0$, we obtain (e_n) is mo-convergent to zero.*

This example demonstrates that mo-convergence does not, in general, imply o-convergence in a semiprime f-algebra without a unit element. However, if (a_α) is an order bounded net, mo-convergence, and o-convergence are the same for semiprime f-algebras, as given in the following proposition.

Proposition 2.3 *Let (b_α) be an order bounded net in a semiprime f-algebra A. Then $b_\alpha \xrightarrow{o} 0$ in A if and only if $b_\alpha \xrightarrow{mo} 0$ in A.*

Proof Let $b_\alpha \xrightarrow{o} 0$ in A, from Remark 1.3 (8), multiplication by an element in f-algebras is order continuous so $|b_\alpha| \cdot a \xrightarrow{o} 0$ for each $a \in A_+$. Conversely, let $b_\alpha \xrightarrow{mo} 0$ in A. Then $|b_\alpha| \cdot a \xrightarrow{o} 0$ in A for each $a \in A_+$ and $|b_\alpha| \cdot a \xrightarrow{o} 0$ in A^δ by Corollary 2.9 in [10]. It is known that for an order bounded net (b_α) in Dedekind completion of A,

$$\sup_\alpha \inf_{\beta \geq \alpha} (b_\beta) \leq \inf_\alpha \sup_{\beta \geq \alpha} (b_\beta)$$

and

$$b_\alpha \xrightarrow{o} 0 \Leftrightarrow 0 = \inf_\alpha \sup_{\beta \geq \alpha} (b_\beta) = \sup_\alpha \inf_{\beta \geq \alpha} (b_\beta).$$

From Remark 1.3, since multiplication by a positive element in f-algebras is order continuous and lattice homomorphism, we have,

$$0 \leq \inf_\alpha \sup_{\beta \geq \alpha} (|b_\alpha| \cdot a) = 0 \Rightarrow a \cdot [\inf_\alpha \sup_{\beta \geq \alpha} (b_\beta)] = 0.$$

Therefore, we get $\inf_\alpha \sup_{\beta \geq \alpha} (b_\beta) = \sup_\alpha \inf_{\beta \geq \alpha} (b_\beta) = 0$ because A is semiprime. Thus, we have $b_\alpha \xrightarrow{o} 0$ in A^δ. Hence $b_\alpha \xrightarrow{o} 0$ in A by Corollary 2.9 in [10]. □

From the previous Proposition, we obtain the following result.

Corollary 2.4 *Let (b_α) be an order bounded net in a semiprime f-algebra A. Then the following statements are equivalent:*

i) (b_α) is mo-convergent to zero;

ii) (b_α) is o-convergent to zero.

We now examine the relationship between *mo*-convergence and *uo*-convergence for an arbitrary net (b_α). We begin by recalling the following definition.

Definition 2.5 *Let A be an Archimedean f-algebra. A is called square-root closed if for any $0 \leq a \in A$ there exists $0 \leq b \in A$ such that $b^2 = a$.*

If A is square-root closed, then A is semiprime. It is also known that the existence of a unit element in A implies that A is square-root closed [17]. The next example shows that, in general, the converse is not true.

Example 2.6 *The collection $C(\mathbb{R})$ of all real continuous functions on $\mathbb{R}$ which vanish at infinity is an f-algebra. $C_\infty(\mathbb{R})$ is square-root closed, but $C_\infty(\mathbb{R})$ does not have a unit element, see Example 3.10 in [17].*

Proposition 2.7 *Let (b_α) be a net in a square-root closed f-algebra A. If $b_\alpha \xrightarrow{mo} 0$ in A, then $b_\alpha \xrightarrow{uo} 0$ in A.*

Proof Let $b_\alpha \xrightarrow{mo} 0$ in A. Therefore $|b_\alpha| \cdot a \xrightarrow{o} 0$ for each $a \in A_+$. On the other hand, it is also known that $|b_\alpha| \wedge a \leq |b_\alpha|$ and $|b_\alpha| \wedge a \leq a$ for all $a \in A_+$ and for each α. Thus, we have $(|b_\alpha| \wedge a)^2 \leq |b_\alpha| \cdot a$. Hence, we have

$$\begin{aligned} 0 \quad &\leq \quad (|b_\alpha| \wedge a)^2 \leq |b_\alpha| \cdot a \xrightarrow{o} 0 \\ &\Longrightarrow \quad (|b_\alpha| \wedge a)^2 \xrightarrow{o} 0 \end{aligned}$$

for each $a \in A_+$. Then there is a net (c_β) in A with $c_\beta \downarrow 0$, and for every β, there exists α_0 such that $(|b_\alpha| \wedge a)^2 \leq c_\beta$ for all $\alpha \geq \alpha_0$. Since A is square-root closed, for any $0 \leq c_\beta \in A$ there exists $0 \leq d_\beta \in A$ such that $d_\beta^2 = c_\beta$. We will first show that (d_β) is decreasing. Let $\beta_1 \leq \beta_2$ for indexes β_1, β_2. Since c_β is decreasing, and by Remark 1.3 (5), we have,

$$\begin{aligned} c_{\beta_2} \quad &\leq \quad c_{\beta_1} \Longrightarrow d_{\beta_2}^2 \leq d_{\beta_1}^2 \\ &\Longrightarrow \quad d_{\beta_2} \leq d_{\beta_1}. \end{aligned}$$

Therefore, (d_β) is decreasing. Now, we will show that the infimum of (d_β) is zero. For every β,

$$\begin{aligned} 0 \quad &\leq \quad a \leq d_\beta \Longrightarrow 0 \leq a^2 \leq d_\beta^2 \\ &\Longrightarrow \quad 0 \leq a^2 \leq \inf(d_\beta^2) = 0 \\ &\Longrightarrow \quad a^2 = 0 \end{aligned}$$

holds. From Proposition 12.11 in [17], it is known that square-root closed f-algebras are semiprime. Then we get $a = 0$. This implies

$$(|b_\alpha| \wedge a)^2 \leq d_\beta^2 \Longrightarrow |b_\alpha| \wedge a \leq d_\beta \downarrow 0$$

for every β, for all $\alpha \geq \alpha_0$. Thus, we have $b_\alpha \xrightarrow{uo} 0$. □

The following example shows that the converse of the above Proposition is not generally true, even if A is a unitary f-algebra.

Example 2.8 *Let $A = l_\infty$ and $(b_n) = n(e_n)$ be a sequence in A . Then the sequence (b_n) is uo-null since every disjoint sequence is uo-null by Corollary 3.6 in [10]. On the other hand, (b_n) is not mo-null.*

From Remark 1.2 (4), we get the following result.

Corollary 2.9 *If A is a unital f-algebra, then uo-convergence and mo-convergence are equivalent if and only if A is finite-dimensional.*

3. Multiplicative order continuous operators

In this section, we investigate the order structure of the space of mo-continuous operators and examine its relationship with o-continuous and uo-continuous operators. The space $L_n(R, F)$ of the o-continuous operators is a band in $L_b(R, F)$ for a Riesz space R and a Dedekind complete Riesz space F.

Definition 3.1 *Let A and B be commutative Riesz algebras. An operator $\Psi : A \to B$ is said to be mo-continuous if $b_\alpha \xrightarrow{mo} 0$ in A implies $\Psi(b_\alpha) \xrightarrow{mo} 0$ in B.*

The definition above was introduced in [7], where several properties of mo-continuous operators were also established. Theorem 2.1 in [1] shows that every o-continuous operator Ψ defined between two Riesz spaces is order bounded. This property does not hold for mo-continuous operators.

Example 3.2 *Consider the operator* $\Psi : L_2[0,1] \to c_0$ *defined by*

$$\Psi(f) = (\int_0^1 f(t)\sin t dt, \int_0^1 f(t)\sin 2t dt, ..., \int_0^1 f(t)\sin nt dt, ...).$$

This operator is norm continuous but not order bounded, while still being mo-continuous. It is easy to show that Ψ *is norm continuous. If we take* $A = \{\sin kt : k \in \mathbb{N}\}$, *it can be seen that* A *is order bounded in* $L_2[0,1]$ *but* $\Psi(A)$ *is not order bounded in* c_0, *so* Ψ *is not order bounded. To see the mo-continuity of* Ψ, *let us show that* $\Psi(f_\alpha)$ *also mo-converges to zero in* c_0 *while* (f_α) *is a net that mo-converges to zero in* $L_2[0,1]$. *If the operator* $\overline{\Psi} : L_2[0,1] \to l_\infty$ *is defined in the form* $\overline{\Psi}(f) = \Psi(f)$, *then the operator* $\overline{\Psi}$ *is norm continuous, and since* l_∞ *is an AM-space with unit,* $\overline{\Psi}$ *is order bounded by 1.5 Theorem in [18]. Moreover,* $\overline{\Psi}$ *is an order to norm continuous operator. If* (g_α) *is a net that order converges to zero in* $L_2[0,1]$, *then* (g_α) *norm converges to zero since* $L_2[0,1]$ *has an order continuous norm. In this case, since* $\overline{\Psi}(g_\alpha)$ *norm converges to zero, it can be seen that* $\overline{\Psi}$ *is order to norm continuous. From Proposition 2.9 in [13],* $\overline{\Psi}$ *is order continuous. Since* (f_α) *is mo-convergent to zero and* $L_2[0,1]$ *has a unit,* (f_α) *order converges to zero, and so* $\overline{\Psi}(f_\alpha)$ *order converges to zero in* l_∞. *Thus, there exists a net* (a_β) *in* l_∞ *with* $a_\beta \downarrow 0$, *and for every* β, *there exists* α_0 *such that* $\left|\overline{\Psi}(f_\alpha)\right| \leq a_\beta$ *for all* $\alpha \geq \alpha_0$. *For all* β *and all* $0 \leq a \in c_0$ *we have* $a_\beta \cdot a \in c_0$ *and* $a_\beta \cdot a \downarrow 0$ *in* l_∞. *From Theorem 1.35 in [2],* $a_\beta \cdot a \downarrow 0$ *in* c_0. *On the other hand, since*

$$|\Psi(f_\alpha)| \cdot a = \left|\overline{\Psi}(f_\alpha)\right| \cdot a \leq a_\beta \cdot a \downarrow 0,$$

we obtain $\Psi(f_\alpha)$ *mo-converges to zero in* c_0.

We will denote the space of order bounded mo-continuous operators as $L_{mo}(R,F)$.

Proposition 3.3 *Let* A *and* B *be* f*-algebras with* B *semiprime. Then* $L_{mo}(A,B) \subseteq L_n(A,B)$.

Proof Let $0 \leq c_\alpha \xrightarrow{o} 0$ in A. Thus, there exists an index α_0 such that $(c_\alpha)_{\alpha \geq \alpha_0}$ an order bounded net and $(\Psi c_\alpha)_{\alpha \geq \alpha_0}$ is also order bounded. Then, we can assume that (c_α) and (Ψc_α) are order bounded. By Proposition 2.3, mo-convergence and order convergence coincide for such nets, and the result follows. □

The converse direction of inclusion is generally not true.

Example 3.4 *Let* $A = c_{00}$, *the space of all real sequences with only finitely many nonzero terms, and let* $B = c_0$, *both equipped with the coordinatewise ordering and coordinatewise multiplication. Consider the embedding operator* $i : A \to B$ *defined by* $i(a) = a$ *for every* $a \in A$. *Since* A *is ideal in* B, *it is regular. Therefore, by Lemma 2.5 in [10], every net* (a_α) *that converges in order in* A *also converges in order in* B, *implying that* i *is order continuous. Now, define a sequence* $(b_n) \subseteq A$ *as follows:*

$$b_1 = (1,0,0,0,...), b_2 = (0,2,0,0,...), ..., b_n = (0,0,...,0,n,0,0,...).$$

For any positive element $a \in A$, $(b_n) \cdot a$ results in a sequence identically zero beyond a particular index. Hence, the sequence (b_n) is mo-convergent to zero in A. However, consider the sequence $a = (\frac{1}{m}) \in B$. In this case, $(b_n) \cdot a$ corresponds to the standard basis (e_n). Since (e_n) does not order converge in B, it follows that (b_n) is not mo-convergent in B. Therefore, the embedding i is not mo-continuous. Thus, $L_n(A,B) \nsubseteq L_{mo}(A,B)$.

Let us give the order structure of $L_{mo}(A,B)$ in $L_b(A,B)$. To do this, we will first obtain Veksler's extension theorem [2, Theorem 1.65] for o-continuous operators for mo-continuous operators.

Theorem 3.5 *Suppose G is a f-subalgebra of a semiprime f-algebra A, and B is a Dedekind complete f-algebra. Assume that $\Psi : G \to B$ is a mo-continuous operator so that $\{|\Psi h| : h \in G \text{ and } 0 \leq h \leq c\}$ is order bounded for any $c \in A_+$. Let the function $\hat{\Psi} : A_+ \to B$ be defined as follows*

$$\hat{\Psi}(c) = \sup\{|\Psi h| : h \in G \text{ and } 0 \leq h \leq c\},\ c \in A_+.$$

If (c_α) is a mo-null net in A_+, $\hat{\Psi}(c_\alpha)$ is a mo-null net in B.

Proof Consider a net (c_α) in A with $0 \leq c_\alpha \xrightarrow{mo} 0$. This implies

$$\begin{aligned} c_\alpha \xrightarrow{mo} 0 &\implies \forall a \in A_+,\ c_\alpha \cdot a \xrightarrow{o} 0 \\ &\implies \forall a \in A_+, \exists (a_\beta) \subseteq A_+, \forall \beta, \exists \alpha_0,\ c_\alpha \cdot a \leq a_\beta \downarrow 0 \text{ for each } \alpha \geq \alpha_0. \end{aligned}$$

Denote the directed index set of the net (c_α) by Λ. We analyze two cases:
Case 1: Λ has a maximal element. Let α_1 be this maximal index, meaning if $\alpha_1 \leq \alpha$ then, $\alpha = \alpha_1$ for all α. Define $a = c_{\alpha_1}$. We have

$$\begin{aligned} c_\alpha \xrightarrow{mo} 0 &\implies c_\alpha \cdot a \xrightarrow{o} 0 \\ &\implies \exists (a_\beta) \subseteq A_+, \forall \beta, \exists \alpha_0,\ c_\alpha \cdot a \leq a_\beta \downarrow 0 \text{ for each } \alpha \geq \alpha_0. \end{aligned}$$

Choosing α_2 such that $\alpha_0 \leq \alpha_2$ and $\alpha_1 \leq \alpha_2$, we observe that for all $\alpha \geq \alpha_2$, we get $c_\alpha = c_{\alpha_1}$, leading to $c_\alpha \cdot c_{\alpha_1} = c_{\alpha_1} \cdot c_{\alpha_1} \leq a_\beta \downarrow 0$, and so $c_{\alpha_1} = 0$. This implies that $\hat{\Psi}(c_\alpha) \cdot b \xrightarrow{o} 0$ for each $b \in B_+$.
Case 2: Λ has no maximal element, we proceed as follows. Define $\Upsilon_\alpha = \{(\alpha, h) : h \in G \text{ and } 0 \leq h \leq c_\alpha\}$ for each α, and set $\Upsilon = \bigcup \Upsilon_\alpha$. We introduce a lexicographic ordering on Υ, setting $(\alpha_1, h_1) \leq (\alpha_2, h_2)$ if either $\alpha_1 < \alpha_2$ or ($\alpha_1 = \alpha_2$ and $h_1 \leq h_2$). This makes Υ a directed upward set. For each $\gamma = (\alpha, h) \in \Upsilon$, define $c_\gamma = h$. Then, we get

$$c_\gamma \cdot a = h \cdot a \leq c_\alpha \cdot a \leq a_\beta$$

for every $\gamma = (\alpha, h) \geq \gamma_0 = (\alpha_0, 0)$ and so $c_\gamma \xrightarrow{mo} 0$ in A. According to Theorem 2.8 in [10], this implies $c_\gamma \xrightarrow{mo} 0$ in G. Since Ψ is a mo-continuous operator on G, and B is Dedekind complete, there exists an index γ_1 and a net (b_γ) in B satisfying $|\Psi c_\gamma| \cdot b \leq b_\gamma \downarrow 0$ for each $\gamma \geq \gamma_1$ and $b \in B_+$. Define $d_\alpha = b_{(\alpha,0)}$ for each α. It is straightforward to verify that $d_\alpha \geq d_{\alpha'}$ is satisfied for every $\alpha \leq \alpha'$. Since Λ does not have a maximal element, Λ has an element $\alpha < \alpha'$ for each α and $b_{(\alpha,h)} \geq b_{(\alpha',0)} = d_{\alpha'}$ for each $0 \leq h \leq c_\alpha$. It follows that

$$0 = \bigwedge_{\gamma \in \Upsilon} b_\gamma = \bigwedge_{\alpha \in \Lambda} \bigwedge_{0 \leq h \leq c_\alpha} b_{(\alpha,h)} \geq \bigwedge_{\alpha \in \Lambda} d_\alpha$$

and so we have $d_\alpha \downarrow 0$. Let us choose α_2 as $\alpha_2 > \alpha_1$. For any $\alpha > \alpha_2$, we have $\gamma = (\alpha, h) \geq (\alpha, 0) \geq \gamma_1$, leading to

$$|\Psi h| \cdot b = |\Psi c_\gamma| \cdot b \leq b_\gamma \leq b_{(\alpha,0)} = d_\alpha$$

for each $0 \leq h \in G \cap [0, c_\alpha]$. Taking the supremum over h, we conclude

$$\hat{\Psi}(c_\alpha) \cdot b \leq d_\alpha.$$

Thus, $\hat{\Psi}(c_\alpha)$ is a mo-null net in B, as required. □

Let A be an Archimedean f-algebra and let A^δ be its Dedekind completion. Then A^δ can be equipped with an f-algebra multiplication that extends the multiplication on A. Moreover, A^δ is semiprime if and only if A is semiprime (see page 67 in [17]).

Corollary 3.6 *Let $\Psi : A \to B$ be a positive mo-continuous operator between semiprime f-algebra A and Dedekind complete f-algebra B. Then Ψ has a unique positive mo-continuous operator extension to all of Dedekind completion of A.*

Proof Given that $\Psi : A \to B$ is a positive mo-continuous operator, we first note from Proposition 3.3 that Ψ is also o-continuous. By applying Veksler's Theorem, the formula

$$S(c) = \sup\{\Psi h : h \in G \text{ and } 0 \leq h \leq c\},\ c \in A^\delta_+.$$

defines a unique positive order continuous operator extension $\hat{\Psi}$ of Ψ to all of A^δ with $\hat{\Psi}_{|A^\delta_+} = S$ and $\hat{\Psi}_{|A} = \Psi$. It follows from Theorem 3.5 that $\hat{\Psi}$ is mo-continuous. □

Corollary 3.7 *Let $\Psi : A \longrightarrow B$ be an order bounded operator between two f-algebras, where A is semiprime and B is Dedekind complete. If Ψ is a mo-continuous operator, then $|\Psi|$ is a mo-continuous operator.*

Proof If $G = A$ is taken in Theorem 3.5, then $\hat{\Psi} = |\Psi|$ on A_+ and so $\hat{\Psi} = |\Psi|$ on A. Therefore mo-continuity of $|\Psi|$ is obtained. □

Corollary 3.8 *Let A be a semiprime f-algebra and B be a Dedekind complete f-algebra. Then $L_{mo}(A, B)$ is an ideal in $L_b(A, B)$.*

Proof From Corollary 3.7, $L_{mo}(A, B)$ is a Riesz subspace of $L_b(A, B)$. Let $S \in L_{mo}(A, B)$ and suppose $|S| \leq |\Psi|$ for some $\Psi \in L_{mo}(A, B)$. Since $|\Psi|$ is mo-continuous, it is easy to see that S is mo-continuous. □

Generally, $L_{mo}(A, B)$ is not a band in $L_b(A, B)$.

Example 3.9 *Let $A = c_{00}$. Let us define*

$$\varphi_k : A \longrightarrow \mathbb{R},\ (a_i) \longrightarrow \varphi_k(a_i) = \sum_{i=1}^{k} a_i$$

φ_k is mo-continuous for each k. If we take $\varphi(a_i) = \sum_{i=1}^{\infty} a_i$, then $0 \leq \varphi$, φ_k and $\varphi_k \uparrow \varphi$, but f is not mo-continuous. Let (e_n) be the standard unit sequence in A. Then the sequence (e_n) is mo-null. On the

other hand, $\varphi(e_n)$ is not mo-null because $\varphi(e_n) = 1$ for each n. Thus we obtain $\varphi \notin L_{mo}(A, \mathbb{R})$. It follows $L_{mo}(A, \mathbb{R})$ is not a band.

Proposition 3.10 *Let A be an f-algebra with unit element and B be a semiprime f-algebra. Then $L_{mo}(A, B) = L_n(A, B)$.*

Proof From Proposition 3.3, $L_{mo}(A, B) \subseteq L_n(A, B)$. On the other hand, let $\Psi \in L_n(A, B)$. Since A is a f-algebra with a unit element, $c_\alpha \xrightarrow{mo} 0$ implies $c_\alpha \xrightarrow{o} 0$. Hence $\Psi(c_\alpha) \xrightarrow{o} 0$, it follows that $\Psi(c_\alpha) \xrightarrow{mo} 0$. Thus, we have $\Psi \in L_{mo}(A, B)$ and hence $L_{mo}(A, B) = L_n(A, B)$. □

When B is Dedekind complete, $L_n(A, B)$ is a band in $L_b(A, B)$, we get the following corollary.

Corollary 3.11 *Let A be an f-algebra with unit element and B be a semiprime Dedekind complete f-algebra. Then $L_{mo}(A, B)$ is a band in $L_b(A, B)$.*

From Example 3.4, we know that an o-continuous operator may not be mo-continuous. In Proposition 3.10, we show that every o-continuous operator is mo-continuous when A is an f-algebra with unit element and B be a semiprime f-algebra. We now introduce an additional condition under which an o-continuous operator becomes mo-continuous.

Proposition 3.12 *Let $\Psi : A \to B$ be an algebra homomorphism between two f-algebras with B semiprime. If Ψ is o-continuous and $I_{\Psi(A)} + (I_{\Psi(A)})^d = B$, then Ψ is mo-continuous (where $I_{\Psi(A)}$ is the order ideal generated by $\Psi(A)$).*

Proof Let Ψ be an o-continuous operator and let $0 \leq c_\alpha \xrightarrow{mo} 0$ in A. We shall show that $|\Psi c_\alpha| \cdot b \xrightarrow{o} 0$ for each $b \in B_+$.

Step 1: Let $0 \leq b$ be an element of $\Psi(A)$. From Remark 1.3 (4) and (7), Ψ is a disjointness preserving operator. Since Ψ is o-continuous, Ψ is order bounded and by Theorem 2.40 in [2] $|\Psi|$ exists and $|\Psi a| = |\Psi||a|$ for all $a \in A$. Thus, there exists $a \in A_+$ such that $|\Psi|(a) = b$. By the assumption, we have

$$\begin{aligned}
c_\alpha \xrightarrow{mo} 0 &\implies c_\alpha \cdot a \xrightarrow{o} 0 \\
&\implies \Psi(c_\alpha \cdot a) \xrightarrow{o} 0 \\
&\implies |\Psi(c_\alpha \cdot a)| \xrightarrow{o} 0 \\
&\implies |\Psi c_\alpha \cdot \Psi a| \xrightarrow{o} 0 \\
&\implies |\Psi c_\alpha| \cdot |\Psi a| \xrightarrow{o} 0 \\
&\implies |\Psi c_\alpha| \cdot |\Psi|(a) \xrightarrow{o} 0 \\
&\implies |\Psi c_\alpha| \cdot b \xrightarrow{o} 0.
\end{aligned}$$

Step 2: Let b be a positive element of the ideal generated by $\Psi(A)$ in B. There exist $n \in \mathbb{N}$, $a_1, a_2, ..., a_n \in A_+$ and $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}_+$ such that $b \leq \sum_{i=1}^n \lambda_i |\Psi a_i|$. Thus, we have

$$0 \leq |\Psi c_\alpha| \cdot b \leq |\Psi c_\alpha| \cdot \sum_{i=1}^n \lambda_i |\Psi a_i| = \sum_{i=1}^n \lambda_i |\Psi c_\alpha| \cdot |\Psi| a_i.$$

By step 1, it is easy to see that $|\Psi c_\alpha| \cdot b \xrightarrow{o} 0$.

Step 3: Let b be a positive element of $B = I_{\Psi(A)} + (I_{\Psi(A)})^d$. Then there exist $0 \leq b_1 \in I_{\Psi(A)}$ and $0 \leq b_2 \in I^d_{\Psi(A)}$ with $b = b_1 + b_2$. Then we get

$$0 \leq |\Psi c_\alpha| \cdot b = |\Psi c_\alpha| \cdot b_1 + |\Psi c_\alpha| \cdot b_2 = |\Psi c_\alpha| \cdot b_1$$

because $|\Psi c_\alpha| \cdot b_2 = 0$. By Step 2, it follows that $|\Psi c_\alpha| \cdot b \xrightarrow{o} 0$. □

Corollary 3.13 *Let $\Psi : A \to B$ be an order bounded surjective algebra homomorphism between two f-algebras with B semiprime. If Ψ is o-continuous, then Ψ is mo-continuous*

Example 3.14 *The operator i in Example 3.4 is not mo-continuous, although it is an o-continuous algebra homomorphism.*

We now recall the following proposition, which establishes the equivalence between uo-continuity and o-continuity.

Proposition 3.15 *([19, Corollary 3.4]) Let $\Psi : E \to F$ be an order bounded disjointness preserving operator between two Riesz spaces. Then Ψ is o-continuous if and only if Ψ is uo-continuous.*

Corollary 3.16 *Let A be an f-algebra with a unit element, and let B be a semiprime f-algebra. Then for any order bounded algebra homomorphism $\Psi : A \to B$, the following statements are equivalent:*

i) Ψ is o-continuous;

ii) Ψ is uo-continuous;

iii) Ψ is mo-continuous.

Proof Since every algebra homomorphism between semiprime f-algebras preserves disjointness, the equivalence of (i) and (ii) follows immediately from Proposition 3.15.

$(i) \Rightarrow (iii)$: Assume that Ψ is o-continuous and let (a_α) be a net that is mo-convergent to zero. Since A has a unity, (a_α) o-converges to zero, and therefore $\Psi(a_\alpha)$ o-converges to zero. Moreover, since every o-convergent net in an f-algebra is mo-convergent, it follows that $\Psi(a_\alpha)$ mo-converges to zero.

$(iii) \Rightarrow (i)$: Assume that Ψ is mo-continuous and that (a_α) o-converges to zero. Then, there exists an index α_0 such that $(a_\alpha)_{\alpha \geq \alpha_0}$ order bounded net, and $(\Psi a_\alpha)_{\alpha \geq \alpha_0}$ is also order bounded. Therefore, by Proposition 2.3, the proof is complete □

Example 3.17 *Let A be an f-algebra and $\pi : A \to A$ be an orthomorphism. Then π is a mo-continuous operator. Therefore, every projection is a mo-continuous operator. Indeed, let $0 \leq c_\alpha \xrightarrow{mo} 0$. Then,*

$$\begin{aligned} c_\alpha \xrightarrow{mo} 0 &\Longrightarrow \forall a \in A^+,\ c_\alpha \cdot a \xrightarrow{o} 0 \\ &\Longrightarrow \forall a \in A^+,\ \pi(c_\alpha \cdot a) \xrightarrow{o} 0 \\ &\Longrightarrow \forall a \in A^+,\ \pi(\pi_a(c_\alpha)) \xrightarrow{o} 0 \\ &\Longrightarrow \forall a \in A^+,\ \pi_a(\pi(c_\alpha)) \xrightarrow{o} 0 \\ &\Longrightarrow \pi c_\alpha \cdot a \xrightarrow{o} 0 \end{aligned}$$

where $\pi_a : A \to A$ is an orthomorphism denoted by $\pi_a(c) = a \cdot c$. Therefore π is mo-continuous.

Conflict of interest

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