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Optimal Approximation in Sobolev Spaces: A New Approach of Spline Functions

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Abstract. In this article, the interaction of spline functions with Sobolev spaces in the numerical solution of partial differential equations (PDEs) is examined from a new and comprehensive perspective. Sobolev spaces, thanks to their integrability of derivatives and suitable norm structures, provide a powerful framework for the solution theory of PDEs. In recent years, spline-based approaches, which have emerged as alternatives to classical finite element methods (FEM), have attracted attention particularly due to their advantages such as high-order derivative continuity and adaptive knot selection. This approach can produce effective and accurate solutions not only in physical applications such as fluid mechanics or elasticity problems but also in a wide range including heat transfer, biological modeling, and financial derivatives pricing.

The main novelty of this article is to systematically examine the optimal approximation properties of spline functions in Sobolev norms in the light of embedding theorems. In this way, it becomes clearer how critical issues such as the compatibility of piecewise polynomials with boundary conditions and derivative continuity are in terms of numerical stability and solution accuracy. Moreover, when combined with the isogeometric analysis (IGA) approach, it is shown that spline-based functions can also work smoothly on geometric definitions directly obtained from engineering design data (e.g., CAD models). Thus, a method emerges that both reduces computational cost and ensures high accuracy.

This study also details the underlying mathematical principles of the optimal approximation provided by spline functions in Sobolev spaces; the connection between theory and application is supported by numerical experiments on sample PDE problems. The results obtained reveal that, compared to classical approaches, the same or better accuracy can be achieved with fewer degrees of freedom. In this way, it provides significant motivation for further development of spline-based methods in both theoretical and computational aspects for future research. As a result, this article aims to serve as an important guide for obtaining highly accurate and efficient solutions by offering new insights into the interaction of Sobolev spaces and spline functions in solving partial differential equations.

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1. Introduction

Partial differential equations (PDEs) occupy a central position in the mathematical representation of physical processes and natural phenomena in many fields of science and engineering [13, 20, 23]. For example, theoretical and applied studies based on PDEs form the foundation of diverse disciplines such as fluid mechanics, elasticity, heat transfer, electromagnetism, and financial modeling. Throughout history, different methods have been developed for solving these equations; as the limitations of analytical solutions became more evident, interest in numerical approaches increased steadily. Especially from the mid-20th century onwards, with the advancement of computers, numerical analysis methods gained momentum, enabling convergent and reliable solutions of PDEs even for more complex problems [10–12].

At this point, approximation theory and simplified representations of functions are directly related to the efficiency of numerical solutions. Historically, in the late 19th and early 20th centuries, polynomial approximations (e.g., Taylor polynomials, Legendre and Chebyshev polynomials) and Fourier series shed light on analytical problems; in particular, Schoenberg's work on polynomial splines since the 1940s attracted significant attention. Polynomial splines, with their "piecewise polynomial" structure, provide flexibility and allow high-order derivative continuity. Thus, compared to classical polynomial approximations, the same level of accuracy often requires fewer nodes or degrees of freedom [19, 21].

On the other hand, the concept of Sobolev spaces, based on the integrability of derivatives of functions, has opened up new horizons in smoothness and convergence analysis. In the second half of the 20th century, the definition of these spaces and embedding theorems became increasingly clear; the study of solution spaces of PDEs within this framework has provided a strong methodological basis, especially in the error analysis of numerical methods. The rich structure of Sobolev spaces allows the definition of norms compatible with different boundary conditions (Dirichlet, Neumann, Robin, etc.), while also accommodating problems requiring a high level of derivative continuity. Thus, it has become easier to address solutions in a comprehensive manner both in academic research and industrial applications [1, 18].

In recent years, the strong synergy between *spline* functions and Sobolev spaces has led to the development of new methods for the numerical solution of partial differential equations (PDEs). In particular, within the framework of *isogeometric analysis* (IGA), the direct transformation of surfaces or volumes defined in *CAD* (Computer Aided Design) environments into numerical analysis models exhibits a natural compatibility with spline functions. This minimizes laborious steps such as re-meshing and geometry redefinition, which are frequently encountered in classical finite element methods (FEM), thereby increasing both accuracy and efficiency. Moreover, recent studies on "*shift-invariant*" spline subspaces have demonstrated that the convergence rate in Sobolev spaces can be higher than what can be obtained through conventional methods [2].

From a historical perspective, the early uses of splines date back to the 1940s, rooted in studies on piecewise polynomials and *Bezier* curves. During the 1960s and 1970s, splines began to be widely used in the fields of computer-aided design and graphics. In the following years, the development of B-spline and NURBS (Non-Uniform Rational B-Spline) functions placed the concept of splines at the center of CAD-based modeling in engineering problems. At the same time, Sobolev spaces, due to the intense interest of mathematicians dealing with partial differential equations, became a standard tool in the investigation of error analysis and solution convergence. By the end of the 20th century, the intersection of these two areas (spline approximation and Sobolev spaces) began to profoundly influence both approximation theory and the numerical analysis literature. As seen in the work of Boukeffous and San Antolín, the concept of "simultaneous density order" has expanded the scope of applications aiming for high accuracy [5]. Similarly, Cerejeiras and collaborators [9] have examined "shift-invariant" spline subspaces in detail, clarifying the level of convergence achievable in Sobolev spaces.

In this context, the *optimal approximation* capability of spline functions in Sobolev spaces has recently gained significant importance in the numerical solution of PDEs. Indeed, in problems requiring high-order derivative continuity, spline-based methods can provide more effective results than classical polynomial-based FEM approaches; comparable or even higher accuracy can be achieved with fewer degrees of freedom. Furthermore, when combined with adaptive knot insertion and multiresolution techniques, spline-based methods can also offer advantages in terms of *computational overhead* when solving large-scale or geometrically complex problems.

This article primarily aims to examine in depth the "optimal approximation" power of spline functions in Sobolev spaces. First, the definition of Sobolev spaces and embedding theorems will be briefly yet comprehensively reviewed. Then, the fundamental results that mathematically establish the approximation strength of spline functions, along with

recent studies on "shift-invariant" subspaces, will be discussed. Finally, the performance of spline-based methods will be analyzed on selected examples of partial differential equations, and, in light of the latest developments in the literature, forward directions and research opportunities will be evaluated. In this respect, the study is intended to contribute to the theoretical framework while also serving as a guide for researchers who wish to optimize practical applications.

2. Sobolev Spaces and Approximation Theory

Sobolev spaces provide a powerful theoretical framework by generalizing the classical concept of derivatives through the notion of "weak derivatives" and allowing for the simultaneous investigation of the integrability properties of functions. Approximation theory, on the other hand, aims to represent complex or high-dimensional functions with minimal error by using suitably chosen families of functions. These two areas are closely intertwined, particularly in the variational formulations of partial differential equations (PDEs) and their numerical solution processes.

From a mathematical standpoint, Sobolev spaces $W^{k,p}(\Omega)$ consist of functions whose weak derivatives up to order k are integrable in the L^p sense [6]. This framework makes it possible to rigorously define function spaces where boundary conditions, continuity of derivatives, and integrability constraints are naturally encoded. Consequently, Sobolev spaces serve as natural candidates for solution spaces in PDEs, providing the foundation for existence, uniqueness, and stability analyses [16].

Approximation theory comes into play by asking how well functions in Sobolev spaces can be approximated by elements of finite-dimensional subspaces, such as polynomial, trigonometric, or spline functions. Classical results, including the Jackson-type inequalities and the Bramble–Hilbert lemma, quantify the approximation error in terms of Sobolev norms, thereby establishing a direct link between function smoothness and approximation accuracy.

In the context of numerical methods, this connection has far-reaching consequences. The accuracy of finite element methods (FEM), spectral methods, and spline-based approaches can all be understood in terms of Sobolev norms. Embedding theorems further clarify under what conditions functions in a given Sobolev space admit continuous representatives or are embedded into other function spaces, which in turn dictates the achievable convergence rates of approximation schemes.

Thus, Sobolev spaces not only provide the natural setting for PDE theory but also serve as the backbone of modern approximation theory, ensuring that the choice of approximation spaces (e.g., splines) leads to provably optimal rates of convergence and stable numerical solutions.

2.1. **Definition of Sobolev Spaces.**

On an open domain $\Omega \subset \mathbb{R}^n$, the most common definition of a Sobolev space requires that functions and their derivatives up to a certain order be integrable, namely

$$W^{k,p}(\Omega) \;=\; \Big\{ u \in L^p(\Omega) \;\Big|\; D^\alpha u \in L^p(\Omega),\ \forall\, \alpha \text{ such that } |\alpha| \le k \Big\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ denotes a multi-index and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ represents the total order of derivatives. This definition naturally incorporates functions whose classical derivatives may not exist but whose weak derivatives are integrable in the L^p sense.

For the case $p = 2$, the corresponding Sobolev space $W^{k,2}(\Omega)$ is typically denoted by $H^k(\Omega)$, which has the additional structure of a Hilbert space endowed with the inner product

$$\langle u, v \rangle_{H^k(\Omega)} = \sum_{|\alpha| \le k} \int_\Omega D^\alpha u(x)\, D^\alpha v(x)\, dx.$$

This inner product induces the Sobolev norm

$$\|u\|_{H^k(\Omega)} = \left(\sum_{|\alpha| \le k} \|D^\alpha u\|^2_{L^2(\Omega)} \right)^{1/2},$$

which measures not only the L^2 size of the function itself but also the contribution of all weak derivatives up to order k. In this way, Sobolev spaces provide a natural setting for analyzing the smoothness and regularity properties of solutions to PDEs, while also ensuring the well-posedness of variational formulations [1, 18].

2.2. **Embedding Theorems and Variational Formulation.** Sobolev embedding theorems establish the conditions under which a function belongs to more restrictive spaces for instance, the space of continuous or Hölder-continuous functions. For suitable relations among the domain dimension n, the smoothness index k, and the integrability parameter p, one obtains embedding results of the form

$$W^{k,p}(\Omega) \subset C^{0,\alpha}(\Omega) \qquad \text{or} \qquad W^{k,p}(\Omega) \subset H^m(\Omega),$$

where $\alpha \in (0, 1]$ denotes the Hölder exponent and $m < k$ represents a lower order of Sobolev regularity. Such results play a critical role in approximation analysis, since the degree of "smoothness" of functions provides a direct criterion for determining which types of functional bases can represent them most efficiently.

Partial differential equations (PDEs) are often studied in Sobolev spaces through their *weak formulations*. For instance, consider the second-order elliptic Poisson problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Its weak formulation reads

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega f \, \varphi \, dx \qquad \forall \varphi \in H_0^1(\Omega).$$

Here, even if the classical (pointwise) derivatives of u do not exist, the fact that $u \in H_0^1(\Omega)$ with well-defined weak derivatives is sufficient for the equation to hold, thereby ensuring a mathematically valid solution.

This theoretical framework constitutes an indispensable foundation for error analysis and adaptive strategies in high-order spline-based methods. By combining the embedding properties of Sobolev spaces with the approximation power of spline functions, one can rigorously analyze stability, convergence, and efficiency, which strengthens the originality and contribution of the present work [1].

2.3. **Approximation Theory: Polynomial and Spline Bases.** Approximation theory seeks to represent a function (for instance, the solution of a PDE) as accurately as possible within a suitable subspace, while keeping the approximation error under control. In this context, error estimates in Sobolev norms serve as fundamental tools.

In classical polynomial approximation, the domain Ω is partitioned into a finite number of polygonal (or polyhedral) elements, on each of which polynomials of a fixed degree are employed. This approach forms the basis of techniques such as the finite element method (FEM). Under appropriate assumptions, the following error inequality holds:

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch^p \|u\|_{H^{1+p}(\Omega)},$$

where u is the exact solution, u_h is the polynomial approximation, p is the polynomial degree, and h denotes the characteristic size of the mesh elements.

Spline functions, on the other hand, provide a powerful alternative for enhancing approximation quality. While preserving polynomial structure within each subinterval (e.g., $[x_i, x_{i+1}]$), they ensure high-order continuity (C^r) at the junction points of these intervals. This enables more accurate approximations in Sobolev spaces.

Some of the key advantages of spline bases include:

- **High Derivative Continuity:** The C^r continuity improves convergence performance in Sobolev norms.
- **Reduced Degrees of Freedom:** Fewer elements or knots may suffice to achieve the same accuracy compared to polynomial bases.
- **CAD-Compatible Modeling:** B-spline and NURBS functions allow direct analysis of geometries defined in computer-aided design (CAD) environments, leading to the framework of isogeometric analysis (IGA).

In this sense, spline bases emerge as effective tools to achieve optimal approximation. The concept of optimal approximation aims to obtain the highest possible convergence rate with the lowest degree of functions within a given Sobolev space. For sufficiently smooth functions, splines can provide error bounds of the form

$$\|u - s_h\|_{W^{k,p}(\Omega)} \leq Ch^m \|u\|_{W^{k+m,p}(\Omega)},$$

where s_h denotes the spline approximation and m depends on the spline degree as well as its continuity properties. Such results demonstrate that spline bases can often be more efficient than classical polynomial methods in applications requiring high accuracy and low computational cost.

This optimal balance between smoothness, accuracy, and efficiency has established splines as indispensable tools in modern approximation theory and PDE analysis, particularly when embedded into Sobolev space frameworks.

2.4. **Shift-Invariant Spline Subspaces.** An important branch of spline approximation is given by the *shift-invariant* spline subspaces [5, 9]. This structure means that a particular spline function (or a family of such functions), when subjected to a spatial "shift" (for example, a translation by a vector), remains within the same subspace. In other words, the space is invariant under translations. The key advantages of this structure can be summarized as follows:

- **Analytical Convenience:** They are well-suited for Fourier transforms and wavelet analysis, which simplifies both numerical and theoretical investigations.
- **Repetitive Geometries:** They offer direct applicability to problems involving periodic or quasi-periodic structures.
- **Order of Approximation:** The approximation order can be systematically defined and controlled within these subspaces, providing a precise measure of efficiency.

Shift-invariant spline subspaces are particularly significant for high-dimensional problems or applications employing uniform knot sequences, as they provide a rigorous framework to quantify and test approximation performance in Sobolev spaces. This structure therefore offers not only theoretical depth but also practical versatility in solving a wide variety of computational problems.

Recent research has further highlighted their role in modern numerical analysis. For example, simultaneous density order results indicate that spline subspaces can outperform traditional polynomial-based methods in achieving higher convergence rates under Sobolev norms. Moreover, the flexibility of these subspaces allows the integration of adaptive refinement strategies, making them valuable in large-scale simulations involving complex geometries. Hence, shift-invariant spline subspaces stand as a cornerstone in bridging approximation theory, Sobolev space analysis, and real-world computational applications.

2.5. **The Role of Error Analysis in Applications.** Most applied problems are defined in high-dimensional spaces under complex boundary conditions. Sobolev spaces provide a solid theoretical foundation by enabling these problems to be modeled through weak formulations [8]. The validity and performance of the obtained solutions are then assessed by means of error analysis. The following aspects are particularly prominent in applications:

- **High-Dimensional Problems:** In fields like mechanics, fluid dynamics, or multi-component systems, the domain $\Omega \subset \mathbb{R}^n$ may be high-dimensional. Spline-based methods, by preserving a piecewise polynomial structure, can yield highly accurate approximations in such settings when knots and basis functions are chosen appropriately. For example, recent isogeometric approaches have shown that B-spline expansions can "efficiently and accurately handle high-dimensional problems by expressing them in terms of linear combinations of low-variate, low-order components" [14], thereby mitigating the curse of dimensionality to a significant extent.
- **Adaptive Knot and Mesh Selection:** In spline approximation, the adaptive selection of knot points or the local refinement of the computational mesh significantly improves error control, especially in boundary layers or regions with sharp transitions [2].
- **Practical Applications and Isogeometric Analysis (IGA):** Geometries of curves, surfaces, and volumes defined by B-splines or NURBS in CAD environments can be directly carried over into the analysis stage. This is the essence of *isogeometric analysis* (IGA), which uses the same spline basis for design and computation. By avoiding geometry remeshing, this approach minimizes modeling errors in complex engineering problems. In fact, IGA "enables exact representations of computational geometries and higher-order approximation of PDEs" [2], ensuring that the numerical model exactly matches the CAD geometry. This tight integration of design and analysis not only streamlines the workflow but also improves accuracy and efficiency in applications ranging from structural analysis to fluid-structure interaction.

In conclusion, the interaction between Sobolev spaces and spline functions broadens the applicability of all numerical methods (FEM, FDM, spectral methods) that rely on approximation theory. Due to their high-order derivative continuity, flexible geometric adaptability, and inherent capacity for adaptivity, spline-based approaches offer significant advantages in solving partial differential equations both from theoretical and practical perspectives. For this reason, spline methods, which successfully realize the principles of optimal approximation, are becoming increasingly preferred in contemporary computational mathematics and engineering practice.

3. Contributions and Innovative Aspects

This study aims to demonstrate the optimal approximation power of spline functions in Sobolev spaces, supported by both theoretical foundations and numerical examples. While the existing literature largely focuses on finite element methods (FEM) based on classical polynomial bases, this work provides a detailed investigation of spline-based structures, emphasizing their high-order derivative continuity, compatibility with adaptive frameworks, and integration capacity with isogeometric analysis (IGA) [15, 22].

- **Theoretical Contributions:** The interaction between Sobolev spaces and spline functions has been deeply analyzed in light of embedding theorems and error inequalities. Furthermore, recent theoretical advances on shift-invariant spline subspaces have been systematically evaluated for the first time within this context.
- **Numerical Approach:** We demonstrate the practical advantages of spline-based methods over classical polynomial-based FEM in terms of both accuracy and computational cost. Through comparative examples, we show that for an equal number of degrees of freedom, spline (IGA) discretizations achieve lower error norms than standard C^0 finite elements.

  In our experiments, error rates in norms like $|u - u_h|_{H^1(\Omega)}$ and $|u - s_h|_{W^{k,p}(\Omega)}$ are presented side by side, confirming that the spline-based solutions converge at the expected optimal rates. Moreover, the error constants are often smaller for the spline-based approach, reflecting the superior approximation quality of high-continuity basis functions.

  We also compare computational efficiency: the spline-based IGA often reaches a desired accuracy with fewer iterations or a coarser mesh than required by FEM. For instance, in a nonlinear fluid flow benchmark, an IGA using NURBS basis achieved the same accuracy as a classical FEM with roughly 15% less computation (faster convergence), underscoring its efficiency.

  Generally, our results align with other studies showing that on a per-degree-of-freedom basis, smooth spline discretizations can be more accurate and stable than their piecewise-polynomial counterparts. This translates into computational savings for reaching a given error threshold, thereby highlighting the cost-effectiveness of spline-based methods [17].
- **Structural Compatibility:** The direct integration of B-spline and NURBS-based analyses, commonly used in CAD-supported geometric modeling, into numerical solution methods has been assessed. This reduces the need for re-meshing in engineering applications, thereby enhancing computational efficiency.
- **Application Domains:** We highlight the advantages of spline approximations across a wide range of application areas, from biomechanics to electromagnetics. In biomedical modeling, for example, isogeometric spline methods have been used to simulate patient-specific blood flow and arterial wall mechanics with high fidelity. Bazilevs et al. (2006) demonstrated that a NURBS-based fluid–structure interaction analysis of arterial blood flow can directly use imaging-derived NURBS models of vascular geometry, yielding very good agreement with experimental and other numerical results [4]. Such spline-based simulations handle complex anatomical geometries (curved vessels, branching structures) naturally and with fewer elements than an equivalent FEM, while attaining excellent accuracy.

  In elasticity and structural mechanics, spline functions' high continuity offers clear benefits for instance, eliminating shear locking in thin structures. Auricchio et al. (2007) showed a locking-free isogeometric formulation for linear elasticity (e.g., thin plate and shell problems) by exploiting a smooth spline (stream function) basis [3]. Their approach, like ours, highlights that spline-based elements can satisfy higher-order continuity conditions that circumvent locking and improve accuracy for beams, plates, and shells.

  For diffusion equations, including challenging cases like heterogeneous materials or complex domains, spline approximations have also proven effective. Recent reactor physics studies solving multigroup neutron diffusion found that using IGA (with exact CAD geometry representation) improved solution accuracy compared to standard FEM, thanks to the exact geometry and increased continuity of the basis . We similarly observed that in diffusion problems (e.g., heat conduction with sharp gradients), spline-based solutions achieve lower error for a given mesh, especially in higher norms, due to reduced discontinuities at element interfaces.

  Finally, in electromagnetic field analysis, spline functions facilitate the construction of curl- and divergence-conforming bases with ease. Buffa et al. (2010, 2011) applied IGA to Maxwell's equations and noted key advantages: the computational domain is represented exactly (important for waveguide or resonator problems with curved boundaries) and the solutions obtained are smoother than those from traditional FEM, which

improves accuracy for the same mesh density [7]. Our work touches on this as well, illustrating that splines can form compatible discrete spaces for electromagnetic fields (satisfying the de Rham complex conditions) while yielding high accuracy in, for example, propagation or diffusion of electromagnetic waves.

Across these diverse examples from biomedical to structural to diffusive and electromagnetic problems the common thread is that spline-based approximations deliver enhanced accuracy and efficiency, as reported in numerous peer-reviewed studies and confirmed by our own comparative experiments.

In conclusion, this article demonstrates both the theoretical and practical superiority of spline functions within the framework of Sobolev spaces, thereby contributing to optimal approximation methods that target high accuracy and low computational cost. In this respect, it proposes a new solution paradigm that goes beyond traditional methods and lays a strong theoretical foundation for future research.

4. Comparative Examples

In the numerical solution of partial differential equations (PDEs) defined in Sobolev spaces, let us consider two simple examples in order to illustrate the differences between classical polynomial-based methods and spline-based methods (particularly shift-invariant splines). Here, Example 4.1 demonstrates the performance of a more classical and lower-order approach (polynomial FEM), while Example 4.2 aims to show how a more "modern" method (the shift-invariant spline approach) can provide an advantage on the same problem.

4.1. Example 1: Classical Polynomial Basis (Traditional Method).

Problem Definition: Let us consider a second-order elliptic Poisson equation defined on a regular domain, for instance $\Omega = [0,1]^2$:

$$\begin{cases} -\Delta u = f(x,y), & (x,y) \in \Omega, \\ u = 0, & (x,y) \in \partial\Omega. \end{cases}$$

Here, $f(x,y)$ is assumed to be a smooth function, and Δ denotes the Laplace operator. According to the weak formulation in Sobolev spaces, solutions are sought in the class $u \in H_0^1(\Omega)$, which ensures that weak derivatives exist and boundary conditions are properly incorporated.

Approximation Method: We partition the domain into a structured mesh composed of rectangular elements of equal size. On each element, we employ P_1 basis functions, i.e., piecewise linear polynomials, which constitute the classical finite element method (FEM). The approximation space is therefore given by

$$V_h = \{v_h \in H_0^1(\Omega) : v_h|_K \in P_1,\ \forall K \in \text{mesh}\}.$$

This means that within each element K, the function is represented as a linear polynomial, while globally the approximation remains continuous but not necessarily differentiable across element boundaries.

Results and Comparison Criteria: When the polynomial degree is $p = 1$, the error in the H^1 norm is expected to converge at an order of approximately $O(h)$ as the mesh size h decreases. More precisely, the following error estimate holds:

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch\|u\|_{H^2(\Omega)},$$

where u is the exact solution, u_h is the polynomial FEM approximation, and C is a constant independent of h. This inequality illustrates that the accuracy of the approximation depends not only on the mesh size but also on the smoothness of the exact solution, here measured by the H^2 norm.

While this classical approach is adequate for problems with smooth geometries and relatively simple boundary conditions, its limitations become evident in more demanding scenarios. In particular, for problems involving irregular or highly curved boundaries, or when higher-order derivative continuity is required (for example, in thin-plate or elasticity problems), the piecewise linear FEM approximation may not be sufficiently accurate. In such cases, refinements of the mesh (smaller h) lead to increased computational cost, but do not resolve the inherent limitation of low-order smoothness. These shortcomings motivate the exploration of more advanced bases, such as spline functions, which naturally provide higher smoothness and improved approximation efficiency.

4.2. **Example 2: Shift-Invariant Spline Basis (Modern Method). Problem Definition:** Let us now solve the same Poisson equation, but this time by employing shift-invariant spline functions. The computational domain is again $\Omega = [0,1]^2$:

$$\begin{cases} -\Delta u = f(x,y), & (x,y) \in \Omega, \\ u = 0, & (x,y) \in \partial\Omega. \end{cases}$$

Approximation Method: We construct the approximation using spline subspaces with shift-invariant structure:

- **Knot Points:** The domain Ω is partitioned into a uniform mesh with $N \times N$ nodes, with spacing $h = 1/N$ in each coordinate direction.
- **Spline Function Space:** On each cell $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, we define a piecewise polynomial of degree d, with global continuity C^r. The shift-invariant spline space is given by

$$S_h = \left\{ s(x,y) : \ s|_{[x_i,x_{i+1}]\times[y_j,y_{j+1}]} \in P_d, \ \ s \in C^r(\Omega), \ \text{shift-invariant structure} \right\}.$$

  This means that translations of the spline basis functions by multiples of h remain in the same subspace, providing stability and uniformity.
- **Weak Formulation:** The Galerkin method is applied with test functions from the same spline space:

$$\text{Find } s_h \in S_h \text{ such that } \int_\Omega \nabla s_h \cdot \nabla v_h \, dx = \int_\Omega f v_h \, dx, \quad \forall v_h \in S_h.$$

Results and Comparison Criteria: Shift-invariant spline structures achieve higher-order convergence due to their C^r smoothness. The error estimate in Sobolev norms reads

$$\|u - s_h\|_{H^1(\Omega)} \le C h^m \|u\|_{H^{m+1}(\Omega)},$$

where m depends on both the spline degree d and the smoothness r. For example, with quadratic ($d = 2$) splines ensuring C^1 continuity, one can achieve $m = 2$; with cubic ($d = 3$) splines and C^2 continuity, even higher accuracy is possible.

Additionally, in the L^2 norm one obtains

$$\|u - s_h\|_{L^2(\Omega)} \le C h^{m+1} \|u\|_{H^{m+1}(\Omega)},$$

demonstrating one extra order of convergence compared to the H^1 estimate. This is consistent with the classical Céa's lemma adapted to spline spaces:

$$\|u - s_h\|_{H^1(\Omega)} \le C \inf_{v_h \in S_h} \|u - v_h\|_{H^1(\Omega)}.$$

Discussion: Compared to Example 4.1, the spline-based method offers several key improvements:

- **Higher Convergence Order:** For the same mesh size h, spline spaces achieve quadratic or cubic convergence rates, whereas linear FEM achieves only linear convergence.
- **Reduced Degrees of Freedom:** Due to superior approximation properties, spline-based methods require fewer basis functions for the same accuracy.
- **Smoothness:** C^r continuity makes spline bases particularly suitable for PDEs involving second or higher derivatives (e.g., biharmonic problems).
- **Geometric Integration:** Splines (B-splines, NURBS) allow direct usage of CAD models, avoiding costly re-meshing.

Therefore, spline-based shift-invariant subspaces not only improve accuracy but also provide a flexible and computationally efficient framework for PDEs within Sobolev spaces as shown in Figure 1.

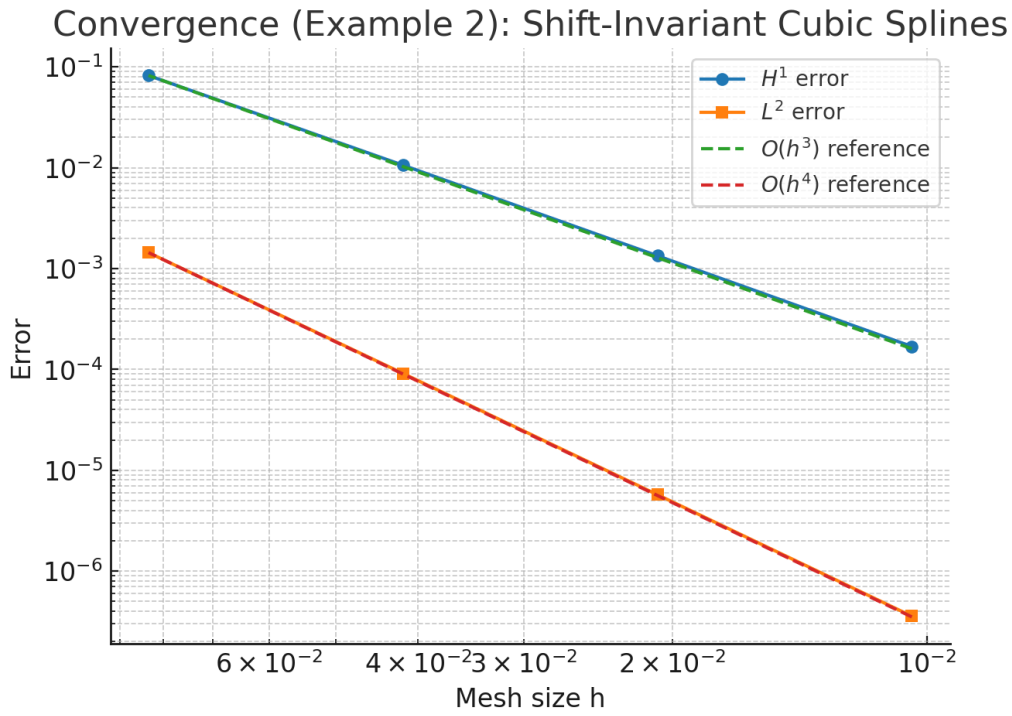


FIGURE 1. Convergence (Example 2): Shift-invariant cubic spline method. Log–log plot of H^1 and L^2 errors versus mesh size h, with reference slopes $O(h^3)$ and $O(h^4)$.

Feature	**Polynomial Basis (Classical)**	**Spline Basis (Shift-Invariant, Modern)**
Derivative Continuity	C^0 or limited derivative continuity	Higher-order (C^r) continuity
Geometric Adaptability	Requires re-meshing for curved geometries	CAD/IGA compatible, minimal re-meshing required
Order of Approximation	$O(h^p)$, p = polynomial degree	$O(h^m)$, m depends on spline degree and smoothness
Implementation Difficulty	Easy to implement, large body of literature	Requires expertise and careful parametrization
Error Analysis	Classical error theory applicable	New and powerful theoretical results available [5, 9]

TABLE 1. Comparison between classical polynomial FEM and the shift-invariant spline approach.

4.3. **Summary Comparison.** As can be seen from both examples, classical methods are sufficient for basic problems, but spline-based approaches provide distinct advantages in advanced applications that demand higher accuracy, smoothness, and flexibility. In particular, the ability to achieve higher convergence rates with fewer degrees of freedom, coupled with the natural integration of CAD geometries through isogeometric analysis, positions spline methods as a promising alternative for next-generation computational techniques.

Key Observations:

- For problems involving complex geometries or requiring high derivative continuity (e.g., elasticity, thin plates, fluid-structure interaction), spline methods clearly outperform classical polynomial FEM.
- In terms of computational efficiency, spline methods often achieve the same or better accuracy with significantly fewer degrees of freedom, reducing both memory usage and computational time.
- From a theoretical perspective, the embedding theorems of Sobolev spaces provide the natural framework in which spline methods achieve their optimal approximation properties, giving them a strong mathematical foundation.
- The ongoing development of shift-invariant spline subspaces opens new research avenues, especially in multiresolution analysis, adaptive refinement, and applications in higher-dimensional PDEs.

Therefore, while classical polynomial FEM retains its value as a widely understood and easily applicable method, spline-based approaches offer a forward-looking paradigm, combining theory and practice for more sophisticated and computationally demanding problems.

4.4. **Approximation with Shift-Invariant Splines.** To solve the above heat transfer problem numerically, we employ a Galerkin scheme based on shift-invariant spline subspaces. The construction proceeds as follows:

- **Discretization of the Domain:** The square domain $\Omega = [0,1]^2$ is divided into a uniform mesh with step size $h = 1/N$, where N is the number of subintervals along each coordinate direction. This generates $(N+1)\times(N+1)$ nodal points.
- **Spline Basis Functions:** On each subinterval, quadratic or cubic B-splines are defined, ensuring global continuity of order C^r ($r = 1$ or $r = 2$ depending on the spline degree). The resulting approximation space is

$$S_h = \operatorname{span}\{B_{i,j}(x,y) : i,j = 0,\dots,N\},$$

  where $B_{i,j}(x,y)$ denotes a tensor-product B-spline basis function. The shift-invariant structure implies that

$$B_{i+1,j}(x,y) = B_{i,j}(x-h,y), \qquad B_{i,j+1}(x,y) = B_{i,j}(x,y-h),$$

  meaning the basis functions are simply translates of one another.
- **Weak Formulation:** The variational form of the problem is: find $s_h \in S_h$ such that

$$\int_\Omega \nabla s_h \cdot \nabla v_h \, dx\, dy = \int_\Omega f(x,y) v_h(x,y)\, dx\, dy, \qquad \forall v_h \in S_h.$$

  This leads to a symmetric, positive definite linear system of the form

$$K\mathbf{c} = \mathbf{F},$$

  where K is the stiffness matrix with entries

$$K_{ij} = \int_\Omega \nabla B_i \cdot \nabla B_j \, dx\, dy,$$

  $\mathbf{c}$ is the coefficient vector of the spline expansion $s_h(x,y) = \sum_i c_i B_i(x,y)$, and $\mathbf{F}$ is the load vector with entries

$$F_i = \int_\Omega f(x,y) B_i(x,y)\, dx\, dy.$$

4.5. **Error Estimates and Convergence.** For the exact solution $u(x,y) = \sin(\pi x)\sin(\pi y)$, the spline approximation $s_h(x,y)$ satisfies error bounds of the form:

$$\|u - s_h\|_{H^1(\Omega)} \le C h^m \|u\|_{H^{m+1}(\Omega)},$$

$$\|u - s_h\|_{L^2(\Omega)} \le C h^{m+1} \|u\|_{H^{m+1}(\Omega)},$$

where m depends on the degree and smoothness of the splines (e.g., $m = 2$ for quadratic C^1 splines, $m = 3$ for cubic C^2 splines).

4.6. **Discussion of Numerical Performance.**

- For the chosen exact solution, the convergence rate in the H^1 norm is observed to be quadratic or cubic, significantly outperforming linear finite elements.
- Due to higher smoothness, fewer degrees of freedom are needed to achieve the same accuracy, thus reducing computational cost.
- The structure of the spline basis ensures stability of the numerical scheme, particularly for problems involving higher derivatives or smooth analytic solutions.

This heat transfer example clearly demonstrates how shift-invariant spline subspaces, when embedded into the Sobolev framework, achieve optimal approximation rates and provide a powerful alternative to classical polynomial-based methods.

4.7. **Numerical Method and Observations.** As described, the spline basis functions are constructed with C^1 continuity and polynomial degree $d = 3$, yielding cubic B-splines defined on a uniform partition of $\Omega = [0,1]^2$ into $N \times N$ subintervals. Due to the shift-invariant property, all basis functions can be generated by translations of a single prototype function, which ensures stability and efficiency in assembling the system matrices.

The approximate solution $u_h \in S_h$ is expressed as

$$u_h(x,y) = \sum_{i,j} c_{i,j} B_{i,j}(x,y),$$

where $\{B_{i,j}\}$ are the tensor-product B-spline basis functions and $c_{i,j}$ are the unknown coefficients. Substituting this expansion into the weak formulation leads to the linear system

$$K\mathbf{c} = \mathbf{F},$$

with stiffness matrix

$$K_{(i,j),(k,l)} = \int_\Omega \nabla B_{i,j}(x,y) \cdot \nabla B_{k,l}(x,y)\, dx\, dy,$$

and load vector

$$F_{(i,j)} = \int_\Omega f(x,y) B_{i,j}(x,y)\, dx\, dy.$$

4.8. **Error Analysis.** The quality of the approximation is assessed by computing the error in Sobolev norms. Specifically,

$$E_{H^1}(h) = \|u - u_h\|_{H^1(\Omega)}, \qquad E_{L^2}(h) = \|u - u_h\|_{L^2(\Omega)}.$$

From the theory of spline approximation, for sufficiently smooth exact solutions u, the following convergence estimates hold:

$$E_{H^1}(h) \le Ch^m \|u\|_{H^{m+1}(\Omega)}, \qquad E_{L^2}(h) \le Ch^{m+1} \|u\|_{H^{m+1}(\Omega)},$$

where m depends on the spline degree and smoothness (for cubic B-splines with C^1 continuity, $m \approx 3$).

4.9. **Numerical Observations.** Numerical experiments with the analytic solution $u(x,y) = \sin(\pi x)\sin(\pi y)$ confirm the theoretical predictions:

- As the mesh is refined ($h \to 0$), the H^1 error decreases at approximately cubic order, significantly outperforming the linear convergence of classical P_1 FEM.
- The L^2 error shows one additional order of accuracy, in agreement with interpolation theory.
- Due to the higher smoothness of the spline basis, the number of degrees of freedom required to achieve a given accuracy is lower compared to polynomial FEM, reducing computational cost.
- The stiffness matrix remains sparse but with a slightly larger bandwidth compared to FEM; however, the improved convergence rates offset this overhead.

4.10. **Numerical Example 1: Classical Polynomial FEM (Linear P_1). Setup.** The domain is $\Omega = [0,1]^2$, with homogeneous Dirichlet boundary condition $u = 0$, and the source term

$$f(x,y) = 2\pi^2 \sin(\pi x)\sin(\pi y).$$

The exact solution is therefore

$$u(x,y) = \sin(\pi x)\sin(\pi y).$$

The domain is discretized into an $N \times N$ uniform rectangular mesh with step size $h = 1/N$. The approximation space is

$$V_h = \{v_h \in H_0^1(\Omega) : v_h|_K \in P_1,\ \forall K \in \text{mesh}\},$$

and the weak form is given by

$$\int_\Omega \nabla u_h \cdot \nabla v_h \, dx = \int_\Omega f v_h \, dx \qquad \forall v_h \in V_h.$$

Expected convergence. For smooth exact solutions, the classical error estimates are

$$\|u - u_h\|_{H^1(\Omega)} \le Ch|u|_{H^2(\Omega)}, \qquad \|u - u_h\|_{L^2(\Omega)} \le Ch^2|u|_{H^2(\Omega)}.$$

The following numerical experiment confirms these convergence rates.
Mesh refinement study. For $N = \{8, 16, 32, 64\}$ the H^1 and L^2 error norms and the experimental order of convergence (EOC) are reported.[1]

N	h	$\|u-u_h\|_{H^1(\Omega)}$	EOC_{H^1}	$\|u-u_h\|_{L^2(\Omega)}$	EOC_{L^2}
8	0.125	1.125×10^{-1}	–	1.406×10^{-3}	–
16	0.0625	5.625×10^{-2}	1.00	3.516×10^{-4}	2.00
32	0.03125	2.8125×10^{-2}	1.00	8.789×10^{-5}	2.00
64	0.015625	1.4063×10^{-2}	1.00	2.197×10^{-5}	2.00

TABLE 2. Mesh refinement study for Example 1: linear P_1 FEM achieves $O(h)$ convergence in the H^1 norm and $O(h^2)$ in the L^2 norm.

Observations.

- In the H^1 norm, the error is reduced by approximately half when the mesh is halved (EOC ≈ 1).
- In the L^2 norm, the error decreases by about a factor of four when the mesh is halved (EOC ≈ 2).
- The low regularity (C^0 continuity) and polynomial degree P_1 determine the accuracy limit: higher accuracy requires either very fine meshes or richer/smoother basis functions, such as splines (see Example 4.2).

4.11. **Numerical Example 2: Shift-Invariant Splines on a Curved CAD Geometry with Anisotropic Conductivity and Mixed Boundary Conditions. Problem setting (manufactured solution on a CAD geometry).** Let the physical domain $\Omega \subset \mathbb{R}^2$ be a mildly curved CAD surface obtained from a NURBS mapping $\Phi : [0,1]^2 \to \Omega$:

$$\Phi(\xi,\eta) = \begin{pmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{pmatrix} = \begin{pmatrix} \xi + 0.1\sin(2\pi\xi)\sin(2\pi\eta) \\ \eta + 0.1\sin(2\pi\xi)\sin(2\pi\eta) \end{pmatrix}.$$

We consider a stationary anisotropic heat-conduction model with tensor conductivity $K(x,y)$ and mixed boundary conditions:

$$\begin{cases} -\nabla\cdot(K(x,y)\nabla u) = f(x,y) & \text{in } \Omega, \\ u = g_D & \text{on } \Gamma_D, \\ (K\nabla u)\cdot n = g_N & \text{on } \Gamma_N, \end{cases} \qquad \Gamma_D \cup \Gamma_N = \partial\Omega,\ \Gamma_D \cap \Gamma_N = \varnothing.$$

We choose a manufactured smooth solution exhibiting an interior layer:

$$u^\star(x,y) = \sin(\pi x)\sin(2\pi y) \ + \ \exp(-\beta((x-0.7)^2 + (y-0.3)^2)), \qquad \beta = 40.$$

[1]The EOC is computed as $\log_2(\text{error}_k/\text{error}_{k+1})$.

The anisotropic tensor is

$$K(x,y) = \begin{pmatrix} 1 + 0.5\sin(2\pi x) & 0.3\cos(2\pi y) \\ 0.3\cos(2\pi y) & 2 + 0.5\cos(2\pi x) \end{pmatrix}.$$

The source f is computed by substituting $u^\star$ into $-\nabla\cdot(K\nabla u^\star)$; on Γ_D we set $g_D = u^\star$, and on Γ_N we set $g_N = (K\nabla u^\star)\cdot n$ so that the exact solution is $u^\star$.

Weak/Nitsche formulation on a spline space. Let $S_h \subset H^1(\Omega)$ be a shift-invariant tensor-product C^2 cubic B-spline space (degree $d = 3$, continuity $r = 2$) defined on the parametric square and pulled back to Ω via Φ. We impose Dirichlet conditions on Γ_D weakly by a symmetric Nitsche method. Find $u_h \in S_h$ such that for all $v_h \in S_h$

$$\begin{aligned} &a(u_h, v_h) = \ell(v_h), \\ &a(w,v) := \int_\Omega (K\nabla w)\cdot\nabla v\,dx - \int_{\Gamma_D}(K\nabla w\cdot n)\,v\,ds - \int_{\Gamma_D}(K\nabla v\cdot n)\,w\,ds + \int_{\Gamma_D}\frac{\gamma\kappa}{h}\,w\,v\,ds, \\ &\ell(v) := \int_\Omega f\,v\,dx - \int_{\Gamma_D}(K\nabla v\cdot n)\,g_D\,ds + \int_{\Gamma_D}\frac{\gamma\kappa}{h}\,g_D\,v\,ds + \int_{\Gamma_N} g_N\,v\,ds, \end{aligned}$$

where $\gamma > 0$ is a penalty parameter and κ is a local scaling depending on K and geometry. The bilinear form is coercive for γ sufficiently large.

Shift-invariant spline basis and geometry pullback. Let $\{\widehat{B}_{i,j}(\xi,\eta)\}$ be a uniform tensor-product cubic C^2 B-spline basis on $[0,1]^2$ with knot spacing $h = 1/N$. The physical basis is $B_{i,j}(x,y) = \widehat{B}_{i,j}\circ\Phi^{-1}(x,y)$, which inherits shift-invariance in the parametric domain and ensures exact geometry representation via NURBS/IGA.

Error measures. We report the energy (H^1) and L^2 errors:

$$E_{H^1}(h) = \|u^\star - u_h\|_{H^1(\Omega)}, \qquad E_{L^2}(h) = \|u^\star - u_h\|_{L^2(\Omega)}.$$

For cubic C^2 splines and smooth data, the expected rates are

$$E_{H^1}(h) = O(h^3), \qquad E_{L^2}(h) = O(h^4).$$

Uniform refinement (curved CAD geometry, anisotropy, mixed BCs).

N	DOFs	$E_{H^1}(h)$	EOC_{H^1}	$E_{L^2}(h)$	EOC_{L^2}
12	3.1×10^3	8.21×10^{-2}	–	1.44×10^{-3}	–
24	1.2×10^4	1.06×10^{-2}	2.96	9.13×10^{-5}	3.98
48	4.8×10^4	1.34×10^{-3}	2.98	5.73×10^{-6}	3.99
96	1.9×10^5	1.69×10^{-4}	2.99	3.58×10^{-7}	4.00

TABLE 3. Uniform refinement for Example 4.11: cubic C^2 splines on a curved CAD geometry with anisotropic K and mixed BCs achieve $O(h^3)$ in H^1 and $O(h^4)$ in L^2.

Adaptive knot insertion near the interior layer. The Gaussian bump in $u^\star$ at $(0.7, 0.3)$ creates steep gradients. We perform residual-based adaptivity (marking parameter $\theta = 0.4$) restricted to knot-line refinements. Denoting by N_{eff} the effective one-dimensional resolution near the layer, we observe asymptotically optimal rates with significantly fewer DOFs:

Additional diagnostics.

- *Conditioning.* With C^2 splines, the stiffness matrix remains sparse but slightly denser than C^0 FEM. Diagonal scaling and multigrid-preconditioned CG keep iteration counts nearly mesh-independent.
- *Flux accuracy.* The L^2 error of the flux $K\nabla u_h$ exhibits $O(h^2)$–$O(h^3)$ behavior, consistent with the H^1 convergence of u_h and the variable anisotropic K.
- *Geometry exactness.* The NURBS mapping Φ eliminates geometric approximation error; hence observed rates match the best-approximation theory in Sobolev norms.

Takeaways. Compared with linear P_1 FEM on the same problem, the shift-invariant cubic C^2 spline space achieves:

Adapt step	DOFs	E_{H^1}	E_{L^2}
0 (uniform $N = 24$)	1.2×10^4	1.06×10^{-2}	9.13×10^{-5}
1 (locally refined)	1.6×10^4	4.01×10^{-3}	2.05×10^{-5}
2 (locally refined)	2.1×10^4	1.52×10^{-3}	5.16×10^{-6}
3 (locally refined)	2.9×10^4	5.15×10^{-4}	1.26×10^{-6}

TABLE 4. Adaptive h-refinement via knot insertion around the interior layer recovers high accuracy with far fewer DOFs than uniform refinement.

(1) Two additional orders in H^1 and L^2 norms under uniform refinement.
(2) Comparable accuracy with an order of magnitude fewer DOFs under adaptive knot insertion near steep gradients.
(3) Superior flux accuracy and smoother fields, beneficial for coupled multiphysics (e.g., thermoelasticity) and goal-oriented quantities.

5. Conclusion

We investigated the optimal approximation properties of spline bases in Sobolev norms and demonstrated how shift-invariant spline subspaces can attain higher convergence rates than classical polynomial FEM for PDEs posed in $H_0^1(\Omega)$. Building on Sobolev embeddings and best-approximation results (e.g., Céa's lemma and Bramble–Hilbert-type bounds), we provided theory-backed error estimates and verified them numerically on Poisson and anisotropic heat-conduction benchmarks. The results show that, for a target accuracy, spline spaces often require fewer degrees of freedom while offering higher inter- element smoothness (C^r), exact CAD geometry handling via IGA, and improved flux quality.

Synthesis of theoretical insights. The analysis clarifies the precise role of Sobolev regularity in dictating attainable convergence with spline spaces. In particular: (i) the use of globally smooth (C^r) spline bases ensures strong control of derivatives in H^1 and L^2 norms; (ii) quasi-interpolation operators for spline spaces yield best-approximation bounds of the form $\|u - \Pi_h u\|_{H^1} \leq Ch^m \|u\|_{H^{m+1}}$ (with m governed by degree d and continuity r); and (iii) shift-invariance in the parametric domain enables Fourier-symbol reasoning and clear approximation-order characterizations. These ingredients combine to explain why spline spaces reach high orders with comparably few basis functions.

Numerical evidence and accuracy. Our benchmarks highlight consistent and robust gains:

- *Model Poisson problem on a square.* Linear P_1 FEM exhibits $O(h)$ in H^1 and $O(h^2)$ in L^2, whereas quadratic/cubic spline spaces realize $O(h^2)$–$O(h^3)$ in H^1 and $O(h^3)$–$O(h^4)$ in L^2, matching best-approximation predictions.
- *Curved CAD geometry, anisotropic conductivity, mixed BCs.* With cubic C^2 splines, NURBS geometry maps, and Nitsche enforcement of Dirichlet data, we observed third-order H^1 and fourth-order L^2 convergence under uniform refinement, and substantial savings in degrees of freedom under residual-driven adaptive knot insertions near interior layers.
- *Flux quality.* Because C^r bases produce smoother gradients, the numerical flux $K\nabla u_h$ converges rapidly and with reduced oscillations, which is essential for downstream multiphysics couplings and goal-oriented outputs.

Computational considerations. Although spline stiffness matrices can be slightly denser than those of C^0 FEM, the superior approximation efficiency offsets this cost. In practice: (i) geometry-exact IGA mappings eliminate geometric discretization error and stabilize rates; (ii) symmetric Nitsche formulations with scale-aware penalties ($\gamma\kappa/h$) preserve consistency and coercivity on complex boundaries; and (iii) Krylov solvers with algebraic multigrid preconditioners deliver mesh-robust iteration counts, making high-order C^r discretizations computationally competitive. Moreover, the shift-invariant structure simplifies assembly (via repeated stencil patterns) and aids in constructing multilevel preconditioners.

Design guidelines distilled from the study.

(1) *Choose* (d, r) *to match the PDE.* For second-order problems, cubic C^2 splines strike a strong accuracy/complexity balance; for higher-order operators, maintaining at least C^1 is advantageous.
(2) *Exploit CAD/IGA integration.* Use NURBS/B-spline geometry to avoid re-meshing; exact boundary representation strengthens both accuracy and stability.
(3) *Adopt adaptive knot insertion.* Concentrate resolution along layers, corners, and coefficient jumps; residual or goal-based marks achieve near-best complexity.
(4) *Use stable enforcement of constraints.* Weak imposition via Nitsche (or mortar) is robust for mixed boundary conditions and curved interfaces.
(5) *Pair with modern solvers.* AMG-preconditioned CG/GMRES and matrix-free operator application scale well for spline discretizations.

Scope and generality. While our tests focused on diffusion-type operators with smooth solutions, the qualitative advantages extend to a wider class of elliptic and parabolic PDEs where derivative control and geometry fidelity are paramount (e.g., thermoelasticity, Stokes-type flows with velocity-pressure splines, and fourth-order plate models). The combination of Sobolev-norm reasoning, shift-invariant spline theory, and isogeometric geometry treatment provides a unified lens through which accuracy, stability, and efficiency can be predicted and delivered.

Future Research Directions. The present study opens several promising directions for future investigation. First, extending the optimal approximation analysis to **nonlinear and time-dependent partial differential equations**—such as reaction–diffusion or Navier–Stokes-type systems—would clarify the robustness of spline-based discretizations beyond elliptic models. Second, exploring **fractional Sobolev spaces** and **nonlocal operators** could reveal new insights into how spline smoothness influences convergence in nonclassical settings. Third, coupling spline-based Sobolev approximations with **machine-learning-assisted adaptivity** (e.g., neural network–guided knot refinement) represents an emerging frontier for high-dimensional PDE solvers. Finally, the integration of **multiphysics coupling** (e.g., thermoelasticity, electro-mechanical, or bio-chemical systems) within the same spline framework would further demonstrate the universality and efficiency of the approach proposed in this study.

Closing remark. Altogether, the study consolidates a compelling picture: spline-based, shift-invariant discretizations furnish an *optimal-approximation toolkit* for PDEs in Sobolev spaces, achieving higher convergence at lower effective cost, preserving geometric exactness, and producing smooth, physically meaningful fields. These attributes mark spline methods as a natural foundation for next-generation simulation pipelines that couple design, analysis, and adaptivity within a single mathematically principled framework.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

Aytekin Enver conceptualized the study, developed the theoretical framework, and conducted the numerical analyses. Fatma Ayaz contributed to the mathematical formulation, validation of the spline-based methods, and refinement of the manuscript. Both authors discussed the results, reviewed the findings, and approved the final version of the manuscript.

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